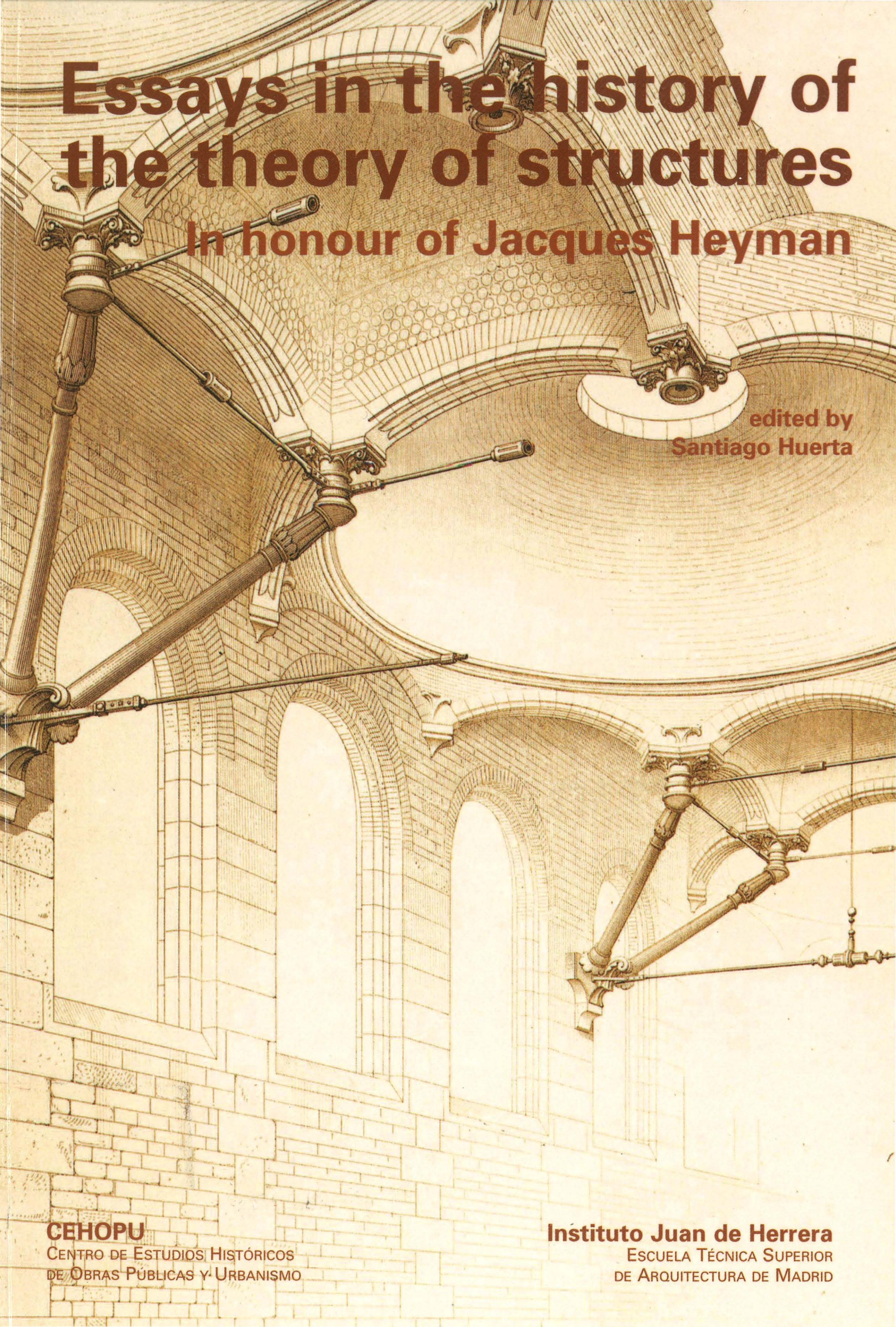
Essays in the history of the theory of structures

In honour of Jacques Heyman

edited by
Santiago Huerta



CEHOPU
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This book is a Festschrift dedicated to Jacques Heyman in the year of his eightieth birthday. Professor Heyman Jacques Heyman has been one of the main contributors to the development of structural theory in the 20th century. He participated crucially in the famous Cambridge Team directed by J. F. Baker, which in the 1940's and 1950's developed the plastic theory of frames, maybe the most important contribution to the science of structural engineering in the 20th century (as elastic analysis was in the previous century). The theory was soon applied, with limitations, to reinforced concrete frames. In the 1960's Jacques Heyman saw that the same theory could be applied to masonry structures and his work added theoretical rigor to a field, which has remained stagnated since the end of the 19th century. Besides, Jacques Heyman proved that this new paradigm may be applied to any "ductile" structure. In fact, the main corollary of the Safe Theorem of limit analysis is, what Heyman calls, "the approach of equilibrium": the analyst may use only two of three fundamental equations, namely, the equations of equilibrium and of the material. The equilibrium approach has been used already by eminent engineers, who followed his structural intuition (Gaudí, Maillart, Nervi,...). This is crucial for a practicing structural engineer or architect, but also for the historians of architecture or engineering, construction historians, who want to go deeper in the understanding of the development of structural forms. Also this last field of the history of construction the work of Jacques Heyman has contributed decisively.

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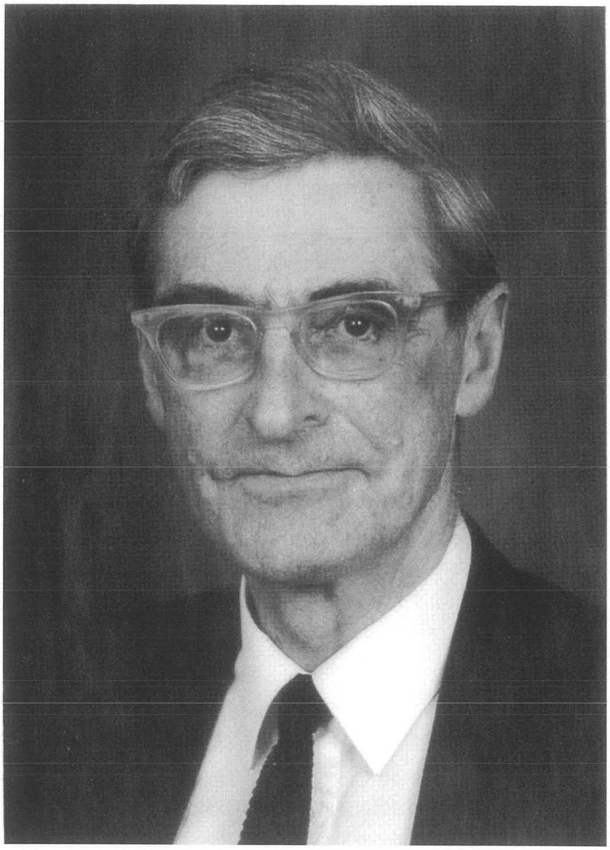

Professor Jacques Heyman

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Foreword by
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Contents

Foreword

The purpose of science is the appreciation of the world formed and modified via the dual procedure of observing and formulating models that become validated or overtaken thanks to new observations. Throughout the 20th century we have witnessed the spectacular substitution of the Newtonian model by the relativist model; the spectacular development of the incredible, expanding universe, and we are now facing the terrible situation in which we can only account for 5% of the energy-matter of the universe, completing the remainder with very unscientific concepts such as “dark matter” and “dark energy,” which overtake the “ether” of the 19th century.

In the field of structures, a subject exclusively technical (how to do things to obtain the required results) since the time of Galileo, we witnessed an early explosion of scientific interest (employing differential calculus), until, towards the end of the 19th century, when there is a consolidated model and a range of equation systems using partial derivatives that exploit the subject from a scientific point of view. This leaves, however, a broad margin for developing technical aspects, both with respect to construction processes and economically viable analysis techniques.

In the 1940’s, a process of refinement of analysis techniques, together with rigorous series of testing, converged in a new model, motivated more by the technical field than by the scientific.

The vision from above of this new model (born, like the elastic model, for steel structures), which endows it with a level of abstraction so high that it far reaches its origin and makes it applicable to any kind of structure and material,

signifies returning the structural problem, if briefly, to the scientific field, which had seemed exhausted since the beginning of the 19th century.

A large part of the praise for transcending the technical aspects and returning to science once more in the field of structural models, belongs to Professor Heyman. Thanks to the early age at which he began making fundamental contributions to the theory of structures, and to his fantastic maturity in which he is still contributing to illuminate the field of structural theory, we are immensely lucky to celebrate with him, in this year of his eightieth birthday, the fantastic generalisations that half a century ago changed the way we understand structures. This change has taken place even though a legion of cultivators of blankets of numbers still make a living from selling complicated models, the accuracy of which lies in the process itself. However, adjusting these models to reality is much more problematic.

RICARDO AROCA HERNÁNDEZ-ROS

Acknowledgements

This book is a Festschrift to Jacques Heyman and gathers the contributions presented in a Symposium also in his honour (Madrid, 2-3 december 2005). We are particularly indebted to all the contributors who have given generously of their time in order to write contributions.

Both the Symposium and the Festschrift have been made possible thanks to the genereous support of many people and institutions. In particular, the support of Professor Aroca, President of the Instituto Juan de Herrera and Dean of the Colegio de Arquitectos de Madrid, was fundamental to make possible both the Symposium and the book. The Director of the School of Architecture, Professor Juan Miguel Hernández de León, has given us all the facilities to organize the Symposium in the buildings of the school. Ms. Amparo Precioso, Gerente of the Centro de Estudios Históricos de Obras Públicas y Urbanismo (Ministerio de Fomento, CEDEX), has supported and encouraged the initiative. Mr. Florentino Santos, Dean of the Colegio de Caminos de Madrid, has facilitated also a generous collaboration. The Associazione Edoardo Benvenuto, and particularly Professors Becchi, Corradi and Foce, have supported and encouraged this initiative from the beginning, as well as Professor John Ochsendorf, who has always been ready to help. The Sociedad Española de Historia de la Construcción has participated with its very modest means. Finally, the Departamento de Estructuras of the School of Architecture has also supported the initiative, helping in secretarial duties as well as with some financial support.

For the organization of the Symposium and editing of the book I have counted on the invaluable collaboration of Ignacio Gil Crespo.

SANTIAGO HUERTA

Introduction
A brief note on the scientific contributions of Jacques Heyman (b. 1925)

Jacques Heyman has been one of the main contributors to the development of structural theory in the 20th century. After graduating he joined the famous Cambridge Team directed by J. F. Baker that established the basis for plastic analysis of steel frame structures. Baker had worked for the Steel Structures Research Committee (SSRC) between 1928 and 1936, with the specific task of writing a rational code for elastic analysis of frames. For the first time, and thanks to the generous financial support coming from the British steel industry, real structures were tested. The results were surprising; elastic analysis was not suitable for predicting the actual state of the structure, contradicting what was believed at the time. Baker soon realised the reason: construction and assembling imperfections, residual stresses accumulated during the rolling process, small foundation movements, etc., led to large changes in the internal stresses of the structure. This failure caused him to change the direction of his research. In 1936 Baker traveled to Germany to attend the second IABSE Conference, where he met those researchers from central Europe who, for over two decades, had been studying the elasto-plastic behaviour of steel beams: Kazincy, Maier-Laibnitz, Bleich, Prager, etc. Back in the UK, he continued studying this field at Bristol University; he repeated Maier-Leibnitz's tests and started a program of tests on frames. The Second World War interrupted the work, which would continue at Cambridge once the war was over.

It is in 1946 when Jacques Heyman, who had recently graduated as an engineer (1944), joined Baker's team. He soon took on great responsibilities in the team and in 1949 he obtained his doctorate degree. He then travelled to the United States to work with William Prager. Professor Prager had escaped from Ger-

many in 1939, and in 1941 he had joined a strong Department of Mathematics at Brown University, where one of his key research topics followed on from the research on plasticity. Baker and Prager established an exchange program between Brown and Cambridge, and this collaboration proved to be crucial. Prager had shown how the three different kinds of statements about mechanics of solids (equilibrium and compatibility equations, and material properties), blended in a single equation in classic elastic theory, are combined in a very different fashion in the field of plasticity. This new approach was proved to be fundamental when time came for rigorously formalising plastic theory for steel frames. The standard duration of post-doctoral exchanges was one year. Jacques Heyman stayed at Brown University for three, and when he returned in 1952 he had acquired a fundamental theoretical base. In 1956 he was co-author (with J. F. Baker and M. Horne) of the first book, rigorously covering plastic theory for the analysis of steel frames: *The steel skeleton. Vol. 2: Plastic behaviour and design*. The book summarised all the work carried out by the Cambridge team over the previous decade and, for the first time in a book about analysis, the Fundamental Theorems of Limit Analysis are stated and applied for design purposes.

These fundamental Theorems had been proven in 1936 by the Russian engineer Gvozdev. However, they had only been published, obscurely, in the Proceedings of the Science Academy of Moscow (in Russian), and they were unnoticed by the international scientific community. The Theorems were rediscovered by Prager's team in the early 1950's. The application of the theorems to the analysis of steel frames allowed for the rigorous use of the plastic analysis that had been carried out in the UK since 1948 (when a clause allowing plastic analysis was added to the British code). In a team effort it is difficult to differentiate individual merits, but it is evident that Professor Heyman occupies a central position in the process of elaboration of the plastic theory applied to steel frames. The list of publications published in this book shows that he took part in most of the key aspects of the theory and the analysis practice. Plastic theory was consolidated in the decade between 1960 and 1970. Specialised studies gave way to the publication of manuals that were crucial for the spread of the theory. They were written by the main figures involved in the process: Beedle, Neal, Horn, etc. Heyman, too, published in 1967 (together with Baker) one of the most comprehensive manuals, a model used for many of the later contributions.

Plastic theory was developed for steel structures and, later, it was seen that it could be applied to reinforced concrete structures. Actually, plastic theory can be applied to any kind of structure that exhibits a ductile behaviour and does not have stability problems. This fact that had been foreseen by some engineers since the beginning of the 20th Century, was clearly and rigorously stated by Jacques Heyman. He is the first one to notice that the Fundamental Theorems meant a

new paradigm that could be applied to all structures built with conventional materials. This could perhaps seem evident for reinforced concrete (in fact, Gvozdev's contribution in 1936 was thought for the analysis of limit loads in reinforced concrete structures). The application in the case of materials such as timber was not as clear and, even less so, the application on stone or brick constructions. Jacques Heyman realised that the Theorems could, also, be translated even for heterogeneous materials such as stone or brick.

In 1966 he published his article "The Stone Skeleton" which constituted a milestone in the development of the modern theory of masonry structures. This highly original and lucid article explains how plastic theory is adapted to the field of masonry architecture. Following a hint from Prager, he realised that, if certain properties are given to the material masonry, the Fundamental Theorems can be translated to suit this case of seemingly different structures. In the field of masonry, over 30 other articles and various books have followed his first article of 1966. In these publications he has applied the modern theory to the study of basic structural elements in masonry buildings (vaults, domes, flying buttresses, towers, spires, etc.).

In fact, his interpretations of Gothic theory close the debates about the structural behaviour of gothic vaults and cathedrals, ongoing since the mid-19th century, occupying the minds of academics such as Viollet-le-Duc, Ungewitter, Mohrmann, Abraham, etc. The deep meaning and the practical consequences of Jacques Heyman's discovery has not been yet really understood by many architects and engineers, who still are using sophisticated computer programs trying to obtain the actual state of internal stresses in masonry buildings.

Finally, we must point out that Jacques Heyman is an internationally recognised scholar in the fields of History of Construction and History of Engineering. He has published memoirs about some of the most important contributions within the history of structural theory. In particular, his book published in 1972 about *Coulomb's memoir on statics* has become a model for these kinds of studies. Also it is worth mentioning his work on specific contributions from other illustrious architects and engineers such as Robert Hooke, Pierre Couplet, Christopher Wren, etc. Within the highly specialised field of the History of Structural Theory (the subject of this book), his work is of crucial importance too. In his book *Structural Analysis, a Historical Approach* (1998), he discusses for the first time the history of the theory of structures, also taking into account the contributions in the field of plastic theory. This completes, therefore, previous attempts by Todhunter and Pearson, Timoshenko, Truesdell and Benvenuto, the main contributors amongst many, who only considered the history of elastic analysis and strength of materials. It is not possible to understand the current situation of structural theory without considering plastic theory.

Jacques Heyman impersonates the best qualities of a humanist engineer, with a great tradition in Spain (Saavedra, Torroja, Fernández Ordóñez, etc.). His competence and originality in theoretical studies of great difficulty have not stopped him from maintaining all throughout his life an interest in history of architecture and engineering, in people (Coulomb, Hooke) and in buildings. In a world where specialisation seems to impose an exclusive dedication to narrow fields, he has shown the importance of general theories when it comes to understanding particular phenomena, and the need of historic studies when it comes to assessing the current situation, even when working on technical fields.

SANTIAGO HUERTA

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Essays in the
History of the Theory of Structures

The History of the Theory of Structures

Jacques Heyman

Not long before his death, Newton wrote: "I do not know what I may appear to the world, but to myself I seem to have been only like a boy playing on the sea-shore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me."

First and foremost, Newton stresses the joy of discovery, of finding something that others have not seen before. For Newton, this might well have been a secret joy, which in fact he kept to himself—he seems to have experienced only weakly the human compulsion to show to others the pearls of wisdom he had found on the beach. The reward for scientists is to know that they have advanced, significantly or even trivially, knowledge of their subject, and there is no absolute compulsion to share this knowledge with others. Indeed, in the seventeenth and eighteenth centuries (and today), there were sometimes good reasons for not revealing too much too soon. When James Bernoulli in 1694 hit on his master "discovery" —that is, a postulate, a hypothesis— that the elastic curvature at any point of an initially straight uniform beam would be proportional to the bending moment applied at that point, he knew that he had found a pearl of great price. He did not reveal at once his discovery for fear of piracy —he wished to establish the theory more firmly before showing others his new smooth pebble.

Second, there is the implication in the words "the sea-shore" and "the great ocean of truth" that all of knowledge is somehow already in existence, and that the business of the scientist is to pick up some pearls and expose them to the light. Advances have indeed been made accidentally, by a scientist noticing a prettier shell than ordinary, and recognizing on examination that it is a find of

some importance. But in most cases, and certainly in Newton's case, the scientist already possesses a few pearls, and has imagined the idea of a beautiful necklace. The string on which the pearls are to be threaded has been conceived to have a certain shape (which may of course change as the work proceeds), and further pearls are sought to match those already discovered, so that the grand design may be achieved.

Thus, in combing the beach beside the great ocean of truth, the scientist is seeking, more or less consciously, for pearls of a particular shape or size. It may be that no such pearls are to be found, and the scientist may then be forced to invent a pearl to thread on to the string —and a totally new discovery will have been made. Of course, in the search on the beach, or arising from a new invention, it may be that something totally unrelated to the quest is discovered. But it is more likely that, in a predetermined and prescribed search, even pearls of very great price indeed will go unregarded if they seem to be irrelevant to the present purpose. Most scientific advances are made in this way; a hypothesis has been proposed and a theory constructed, and only data are collected which support that theory. There is nothing fraudulent in such an approach —if data are found which disprove the theory, then they are welcomed by the true scientist, since a real advance will have been made. As is now axiomatic, nothing can ever be proved to be true, but a single observation may be all that is necessary to prove that something is false. It is not that the scientist suppresses "negative" data, but merely that the search process is predisposed to find *only* pearls which thread well on to the string. It would be totally perverse for scientists to spend their time in seeking to disprove the theories they wish to establish.

* * * *

Historians, like scientists, are rewarded by the joy of discovery. Their field, however, is no longer the virtually infinite ocean of truth explored by scientists, but the more restricted, although still vast, record of human experience. The historian seeks to impose some order, some meaning, on the series of events as recorded by contemporaries, as interpreted by other scholars, and as forcing a pattern on future events. The threads already exist, and must be disentangled and woven into a connected whole. The history of the Great War of 1914–18, for example, must be based on the known events, and interpreted in the light of the intentions of the participants —the forging and breaking of alliances, the politics of the various powers involved, the economic stability of individual countries, not to mention the attitudes of individuals actually engaged in fighting. The threads are very numerous, and may extend a long way— it might be relevant, in discussing World War I, to refer to the conflict between Athens and Sparta of 431 BC, and to analyse, more generally, the conflict between nations that have arisen throughout history.

Historians, like scientists, will select, consciously or not, pieces of evidence to support the pattern they have imposed on their work. Both have in mind a particular shape of necklace for which the pearls are collected. Both may be swayed by the activities of other scholars, by knowledge already at their disposal, even by the aesthetics of their research, but the scientist is not subject to one major constraint imposed on the historian. The historian cannot invent new pearls, but must work with those found lying on the sea-shore. The historian knows which allied countries were victorious in the 1914–18 war (although whether that victory was total may be a subject for separate discussion in the light of the later history of the twentieth century), and whatever account is given of the Great War, the end result must be a known victory in 1918. The historian's evidence will impose itself by its relevance to the creation of this end result, and, as for the scientist, counter evidence may be unwittingly ignored. This is a primitive manifestation of Whig history, in which the evidence adduced is shown to lead inevitably to the known end.

* * * *

The historian of science is a fully qualified Whig historian. In tracing the development of a particular scientific theory, all the evidence is seen as contributing to the progress of human thought from primeval darkness to the present state of illumination. Discoveries not finally relevant to this progress are interpreted as excursions down blind alleys. Thus historians of structural theory know that powerful computer programs exist which determine the state of a great range of structures. Moreover they know that those computer programs make use, in one way or another, of the three kinds of statement that may be made about any structure: There must be equilibrium, static or dynamic, so that relevant equations are involved which connect external and internal forces; geometrical ideas will link internal deformations of the structure to certain boundary conditions on displacement; and known material properties express internal forces in terms of internal deformations. In the search on the sea-shore of the ocean of structural truth, the historian will gather evidence which contributes to the development and understanding of these three master statements of structural theory.

The historian's search has already been narrowed by such a "Whig" approach. The purpose of structural analysis is to provide a basis for structural design, and large buildings have existed for several millennia. Greek and Roman temples were all "designed", as were great Byzantine churches and Romanesque and Gothic cathedrals, but not by modern methods. The search for evidence which leads to an understanding of present-day structural theory will contribute little to an understanding of ancient practice. Ancient design rules were directed, correctly, to the achievement of the proper shape of a structure and its elements; such

numerical rules of proportion will be seen by the historian as not relevant to the present day. (Indeed, some major contributions to mathematics, excursions down blind alleys such as the understanding of irrational numbers, will not be appreciated as arising from building technology).

The historian of modern structural theory will probably start with Galileo's problem (1638) of the breaking of a cantilever beam, which seems to involve the analysis of stress under bending of a rectangular section. But this is to view the matter with modern eyes —Timoshenko, for example, "knows" the "correct" Coulomb solution of 1773, and states that Galileo is in error by a factor of 3. But Galileo was not trying to calculate stresses, but to evaluate a breaking load, and ratios of breaking strengths of different sizes of rectangular beams were calculated correctly by him. In the same way, Coulomb sketches what seems to modern eyes to be an elastic distribution of stress, but was in fact an assumed distribution at fracture; nevertheless, the orthodox elastic solution is credited to Coulomb/Navier, although it was indeed Navier who crystallized formal elastic ideas in 1826. These studies are, as a matter of fact, wrongly classified if they are thought to deal with the theory of structural analysis —they are really an examination of local stresses, and form part of the discipline of strength of materials. Galileo, and Coulomb, studied the statically determinate cantilever, where local failure implies overall structural failure; the theory of structures proper deals with the basic problem of statical indeterminacy.

* * * *

That statically indeterminate, or hyperstatic, structures exist has been known for nearly three hundred years. Various attempts were made throughout the eighteenth century to determine the strength of such structures, but little progress was made. It was appreciated that the equations of statics provided insufficient information, but the continuous beam, for example, could be analyzed only in particular cases. No general theory was available. It was Navier's genius which enabled him to select, from all the pebbles on the sea-shore, just those which could solve the problem. The solution, once formulated, was so compellingly true that subsequent evidence that it was false was largely ignored.

Navier in 1826 was heir to the full Bernoulli/Euler theory of elastic bending. He added the bending equations to those of statics (two of the three master statements) to calculate the complete actual state of a hyperstatic beam. Moreover, once that state was known, the science of strength of materials could be used to determine the values of stress at every point of the beam. Elastic theory was paramount —it was the thread on which all the pearls were mounted. Elastic equations determined the overall state of the structure; elastic analysis determined the greatest values of internal stress, and these were not allowed to exceed

the (experimentally-determined) limiting values (for fracture or gross deformation).

The Navier methodology involves the use of the third master statement of the theory of structures —the boundary conditions describe the way the structure is attached to the environment. It was to be nearly a hundred years before questions were asked about these boundary conditions. Kacinczy in 1914 tried to determine experimentally the degree of fixity needed at the ends of a supposedly fixed-ended beam, not to attack the Navier theory, but to enable a practical design to be made for the end clamping. The experimental result, that the degree of end clamping was really not relevant, triggered the whole development of plastic, as opposed to elastic, design; but, such was the power and beauty of the Navier theory, the (correct) conclusion that elastic theory did not describe the actual behaviour of a hyperstatic structure was not drawn. Indeed, twenty years later, when Kacinczy was fully alive to plastic ideas, he was still mesmerized by Navier, and suggested that a primary elastic solution should be modified to give the required plastic collapse analysis.

A similar fixation on the correctness of elastic concepts is evident a half-century earlier in the work of Castigliano. His energetic principles allowed the determination of internal forces in a hyperstatic structure, and he showed how to calculate the large variations in internal stress which result from very small imperfections in manufacture and construction. But he nowhere remarked that *all* structures will inevitably be imperfect, and moreover imperfect in unknown ways, so that the results of a primary elastic analysis can never actually be observed. It was only in the 1930's that tests in London, on the large steel frames of buildings then under construction, showed conclusively that a simple elastic theory led to a computed stress state far removed from that measured. Even then the investigators sought to modify standard elastic design methods to allow for the "real" behaviour of connexions between members, before finally abandoning over a century of painstaking development in favour of plastic ideas.

Elastic design methods had indeed been developed with great ingenuity. The equations could be written easily, but they were very numerous and, before the advent of the electronic computer, difficult to solve. By contrast the equations of plastic theory were simpler, since (unknown and unknowable) boundary conditions or (for example) flexibility of connexions did not enter the analysis. For a time after World War II the new theory made steady headway, but progress was halted by the increasing use of fast computers. The numerous and complex elastic equations of the nineteenth century could now be solved with great accuracy, and Navier's elastic necklace was allowed to shine with all its undoubted brilliance.

Once again structural designers were dazzled by the power of elastic methods, and continued to believe that their calculations referred to actual observable

states of real constructions. A single counter example should have been all that was necessary to cause the design methods to be modified or indeed abandoned, and by the end of the twentieth century there were countless examples of practical observations which contradicted the theory. It is perhaps forgivable for someone developing a theory to ignore, unwittingly, those pearls which do not fit, usefully and aesthetically, into the necklace under construction; it is less forgivable, once the necklace is complete, to ignore glaring and ugly anomalies.

* * * *

The historian of structural analysis, starting from the Whig standpoint of the present day, when the main features of a correct structural calculation are believed to be known, can trace back the threads and determine where mistakes may have been made. The evidence points inexorably to Navier and the year 1826, when he published the lessons he was giving at the *Ecole des Ponts et Chaussées*. Navier's advance was very great —not perhaps as great as that of Newton in establishing the theory of universal gravitation, but of the same kind; a known and seemingly intractable problem had been solved by a stroke of genius, based on deep thought and a total scholarly grasp of the subject. One difference between the theories of Newton and of Navier was that Newton's theory was continually tested, not in an adversarial way, but as helping to make new advances. And each advance confirmed the correctness of Newton's thought —at least for over two centuries, until crucial small anomalies could be explained by the newer and better theory of Einstein.

Navier's theory was not tested in this way for over a century; that is, experimental confirmation was not sought. Rather, the method was developed by opening up the possibility of the analysis and design of ever more complex structures, and by introducing sophisticated mathematics and elegant ways of making approximate solutions. It is true that some early attempts —even before Navier— were made to test real structures, but the work was difficult and complicated. If the results differed from theoretical prediction, then the usual conclusion was that experimental technique was faulty rather than that the theory was wrong. With hindsight, it would seem that there was an almost deliberate unwillingness to believe anything that might discredit Navier's gigantic intellectual structure.

It is perhaps more true to say that there was no interest in discrediting Navier, who had given the engineer a theory which worked —worked in the sense that structures could be designed. The theory may not be one that describes how a structure actually behaves, but it gives a way of assigning structural dimensions to a building required to carry given loads. Moreover, although the method is in the final analysis irrational, it is safe— internal forces are not in practice those calculated by the elastic theory, but they are forces with which the structure is comfortable,

and which will not cause collapse. This quality of safety was proved only a century and a half after Navier, and, paradoxically, by the application of plastic theorems to the elastic process, but even before this proof elastic designers knew, by experience if nothing else, that their activities were safe. They had no reason to discredit the method they were using and which they believed to be true.

What is exposed here is a fundamental difference between the activities of the "pure" scientist and the applied scientist, that is, the engineer. The engineer develops a scientific theory to the point at which it becomes useful, to the point at which something can be done —in the case of the structural engineer, a building can be designed. Once this point is reached, there is no pressing need to question the validity, the "truth", of the design algorithm that is being used. The "Code of Practice" in which the rules are collected becomes frozen, and continued applications to successful designs confirm its correctness. It is only when the Code is applied to a structure which appears to be typical, but is in fact slightly unusual, that a disaster may occur. Such a disaster should be the single counter-example needed to invalidate the whole theory; more usually, the existing intellectual framework will not be challenged, but some rules of the Code will be slightly modified. In reality this accommodation to unwelcome evidence may be excused —structural engineers are perfectly happy with the mechanics of Newton, and do not need a knowledge of Einstein to design their structures. However, it is the duty of the academic engineer not to ignore inconvenient anomalies, but to try and construct a new and better theory.

Such a new theory must of course retain much of the knowledge accumulated in the past; in the case of structural design, there is no escape from the three master equations —it is the way that those equations are regarded and used that is important. Engineers developing a new theory must continually apply its findings to cases which are indeed unusual; they must seek actively for a counter-example which may lead to a radical revision of the whole intellectual construction.

* * * *

The question arises as to how much of the history of the theory of structures should be taught in engineering schools. Indeed, it could be asked how much theory should be taught, let alone the history of the development of that theory. It is fairly clear that most structural engineers, with a basic but rudimentary grasp of their discipline, and a good knowledge of the building codes, will produce completely satisfactory designs. Universities, by their very nature, concentrate on the teaching of academic theory, of course enlightened by practical experience, but at a level which will prove to be of little use to the graduating engineer. And, although the subject may be of passing interest to a lively student, the history of

engineering will be of even less use. For the intending practical engineer there is little point in mastering the history of the development of structural theory over three centuries, although (for example) the brilliance of Euler's solution of the problem of the elastica will enliven in passing an exposition of the simple theory of buckling. Such individual pearls may be allowed, rightly, to shine, but the practical engineer is really not interested in how the formulas in his Code were derived —the important thing is that the formulas work and allow a design to be made.

It is when formulas do not work that the historian of engineering may have something to contribute. Disasters have occurred through the application of Codes to structures for which they were not intended —examples are the collapse of prefabricated high-rise buildings and of steel box-girder bridges. With a knowledge of the development of the theory and of the way in which rules in the Code were formulated, the historian may be able to determine when and where mistakes were made. This should be the aim of the historian —not to comment on the undoubted size and beauty of the pearls but, on the contrary, to discover that one of the pearls is false, a counterfeit pretending to be part of a perfect pattern, but in reality causing the rupture of the whole necklace. The historian of science is in this sense an anti-scientist; the scientist seeks to advance by finding new pearls, while the historian's aim is destructive.

Students must somehow be made aware that their rules for design may contain a (literally) fatal flaw. It may be that throughout their working lives structural engineers will not design a single building that is unsafe —equally, should the rules be applied to a structure slightly outside the usual, the designer should immediately be on the alert. The student must somehow be convinced that the seemingly perfect and immutable rules that are used for design must, just occasionally, be questioned. Conviction of this sort can be created through a knowledge of past errors, and the analysis of disasters may be the most fruitful area for the teaching of the history of structural engineering.

A history of using scale models to inform the design of structures

Bill Addis

Many engineers today are using the phrase ‘to model a structure’ to describe the process formerly widely known as “structural analysis”. To some extent this has come about as the result of using computers to perform the mathematical operations. First, the engineer has to provide the computer with a spatial arrangement of the structural elements and the nature of the structural connections between them. To this spatial model must be added a model of the loads the structure has to carry. Finally, the computer must be provided with models of each of the materials of which the structure ll be made —their physical properties ranging from density and stiffness to, perhaps, their variation of yield strength with temperature. These three types of mathematical model can then be made to interact in a manner which is presumed to be an accurate representation or model of how the real-world equivalents of the mathematical models would behave (Fig. 1).

The nature of this process of modelling structures and structural behaviour in a computer has its origins in the use of physical models used in the design of structures and, indeed, many other engineering artefacts such as weapons and ships. Models have been used by designers since the very dawn of engineering.

The process of designing a structure has just two elements:

– To provide *dimensions* and materials *specifications* to enable a contractor to begin construction

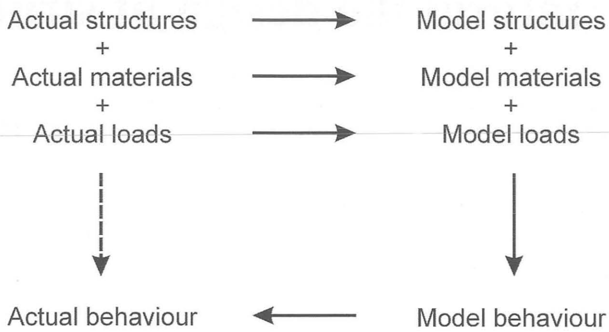

Figure 1
The relationships between the behaviour of full-size and model structures, materials and loads

– To raise the level of *confidence* of the engineer or builder, and especially the client, that a proposed structure *can* be constructed and, once constructed, will work as intended.

The purpose of making scale models of structures is to help the designer discharge these two elements of the design process.

We can divide scale models of structures into two kinds —geometric and structural. Geometric models of unbuilt artefacts were, and still are, used to communicate its appearance and sometimes to help work out how the components of a structure, such as a complex timber truss, might fit together. For such models, the scale is not significant. There is a direct, linear correspondence between the model and the full size artefact.

The purpose of a structural scale model is to raise the engineer's confidence that the structural will work and carry the loads safely. They would be made when the proposed structure is outside the experience of the engineer or builder and when it is too large to test at full scale, either because of the risks involved or the costs. Generally speaking this was when the scale of the proposed structure and, hence, the forces in the full-size structure, would be greater than human scale.

A model is a representation of something —an object or an idea or a type of behaviour, and the nature of the model and the test depends upon its purpose, for example:

- Stability:

  – demonstrate the stability of a masonry structure such as an arch or vault using a model made of individual blocks of wood or plaster

- Elastic behaviour under load:
  - Loading a structure to measure deflections, per se
  - Loading a structure to measure deflections, from which to calculate internal stresses
- Strength of a structure under load:
  - Loading a structure to cause failure of a material in bending, tension or shear
  - Loading a structure to cause failure by buckling (instability)
- Form-finding:
  - create a catenary to find the shape of a suspension structure
  - create a catenary to find the best shape, inverted, for a masonry arch or dome
  - use a hanging sheet to find the form, inverted, of a concrete shell
  - use an elastic sheet to find the shape of a tensile membrane structure
- Construction method or sequence:
  - Devising how to lift and manoeuvre a heavy component into position
  - Checking that a structure is stable at every stage of an unusual erection procedure
  - Demonstrating to a workforce a complex assembly process or sequence of operations
  - Confirming a proposed sequence of stressing a prestressed structure

Making the scale model is only the first stage of the process. The model is then subjected to certain loads and the behaviour of the model is observed —its deformation and ultimately, perhaps, its collapse or the failure of one of its members.

Just as with a virtual model of a structure in a computer, there are four aspects to making the scale model:

- Its geometry —the size and disposition of members / components
- The nature of structural connections, including boundary conditions
- The model loads
- Models of the structural materials.

In each case the relationship between the model and the full-size artefact and the real world need to be chosen, for example:

- Are all dimensions reduced by the same scale factor? Reducing to scale a thin shell or membrane, and bolts or reinforcing rods may be impractical;
- While a real connection may be welded or riveted, it may be more appropriate to model it as pin-jointed; a support on a foundation might be modelled using a spring.
- How to model the self-weight of the structure will depend on the density of the model material; wind loads may be modelled using weights;
- If a steel model is made of a steel truss, the model will be disproportionately stiff in relation to its scale; whereas a model with stiffness reduced to scale may make the model too flexible to handle.

From these observations, it can be seen that the relationship between a scale model of a structure and the full-size structure is more complex than for geometric models:

- Different physical and structural properties vary in different ways with the scale factor. Hence,
- it is not possible to make a single scale model structure that represents the full-size structure and its behaviour in every respect. Hence,
- before making and testing a scale model, the purpose of the model and the test must be decided in order to select the precise ways in which the model does represent the full-size structure. The precise effect of scale on the behaviour must also be understood.

The key question about a model test is thus: What is the relationship between the behaviour of the model and the behaviour of the full-size structure?

Engineers planning the design and construction of a building use scale models to obtain results that will help them solve their immediate problem —how to gain sufficient confidence to construct that particular building.

Scientists, on the other hand, use tests on scale models to help them understand the general laws of nature that govern the behaviour of all structures, at any scale. Put another way, the scientists use scale models to understand the effects of scale on the behaviour of geometrically similar structures.

As well as having different aims, practising engineers and engineering scientists may use different types of scale model and may use the models in different ways (Addis 1988). When developing models to suit their purposes, practising engineers and engineering scientists will apply different criteria as to their suitability, for example:

Criteria for models for practical engineering	*Criteria for models for engineering science*
• Simplicity • Speed of use / cost-effectiveness • Reliability • Degree of approximation • Power to raise confidence in a proposed design	• Accuracy • Generality • Elegance • Ability to account for / explain known phenomena • Ability to predict the outcome of new tests

The history of building and testing scale models of structures is, then, the parallel history of using models to solve particular problems, and to better understand the universal truths about the world —the parallel worlds of the practising engineer and the engineering scientist.

This leads us to what might be called the *paradox of structural model testing*. It is only possible to deal adequately with scale effects in quantitative terms when the theoretical model of the system has been developed. By this time, however, the model test is no longer required.

The paradox is only resolved when we note that the model testing is important for two reasons. First, to help define the mathematical models of structure, loads and materials, which is usually an iterative process. Secondly, as the means of testing the validity and reliability of the theoretical (mathematical) models by comparing theoretical predictions and measured results. In this way the designer's confidence in using the mathematical models is raised sufficiently to use them on the real structure which may be 50–100 times the size of the model (Fig. 2).

The key issue here is that all mathematical models are approximations of the real world. The design engineer needs to know *how* approximate they are, if only for reasons of cost —a sophisticated mathematical analysis or even computer model can be very time-consuming and expensive. Making and testing a physical model can give design engineers the confidence they need that an approximate mathematical model will be accurate enough.

We touch here, upon a more general theme —the nature of progress in engineering. A common view is that progress in practical engineering is achieved by the application of the theoretical ideas developed by engineering scientists. But this simplistic view cannot be true, for there was much progress in practical engineering long before engineering science became established in its modern form (roughly during the century and a half between 1650 and 1800). Rather the interaction between the worlds of the practising engineer and the scientist has

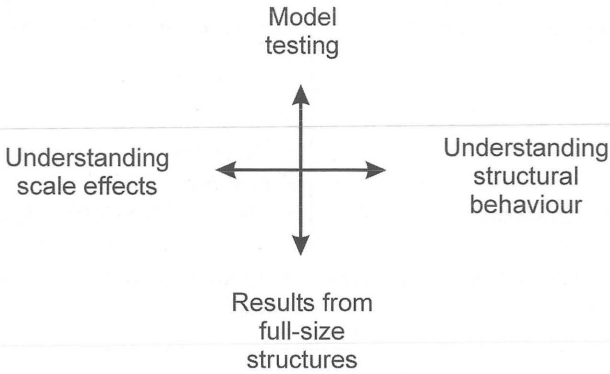

Figure 2
The synergy between understanding structural behaviour and understanding the effects of scale.

provided a fruitful cross-fertilisation, and the making and testing of scale models has formed a central area of common ground. The testing of scale models has been the primary means by which progress has been achieved in both practical engineering construction and engineering science.

The history of using models in design of structures can be separated into three main strands:

- Linear proportion —scale-independent models
- Non-linear relationships —scale-dependent models
- Use of dimensionless constants

Linear proportion. Scale-independent models

The idea that two linked properties vary in direct proportion is born out in our everyday lives —double the volume of a material and its weight is doubled; double the load on a beam and its deflection is doubled. Similarly with the relationship between the dimensions of a two-dimensional drawing or a three-dimensional scale model of a building and those of the full-size building.

Masonry / compression structures

Ancient Greek builders used physical scale models to show their proposed buildings to their patrons. They also devised design procedures, based on linear proportion, to establish all the dimensions of a temple (for example) from a single

key dimension —known as the module. By simply choosing a smaller or larger module, so the scale of the whole temple would be correspondingly reduced or increased. Such procedures for temples in the Doric and Ionic styles are recorded by Vitruvius (c. 25 BC) by which time they had been in use for some 500 years (Addis 1970).

While this approach would have worked for a wide range of sizes for walls and columns, it would not have been satisfactory for beams, whose strength does not vary linearly with scale. The Greeks dealt with this limitation by having an upper limit for the span of stone beams (around 7 metres). If the scale of a temple would have required a larger span, the number of columns within the width or length of the temple was increased from, say, hexastyle (six columns / five spans) to octostyle (eight columns). A similar approach seems to have been taken by Roman builders, though we have no Roman design manuals.

In mediaeval times design procedures were more usually based on sequences of geometric operations, rather than numerical, which had the same effect that a buildings and their elements could be scaled up or down in linear proportion. As in classical times, the dimensions of structural elements whose structural behaviour does not vary in direct proportion to scale —the spans of beams and slenderness of columns— were limited to what was known to work at full scale.

In some ways it might be argued that the builder of masonry structures was lucky in that their stability is independent of scale. If a small model of an arch, flying buttress or ribbed vault was stable, then so would a full-size version ten or twenty times larger. While not understood in terms of modern statics, this phenomenon must have been well known among masons —indeed, it must have been considered entirely obvious you could use a scale model to try out a certain form of arch or vault. Turning this observation around, we can propose that compression structures progressed to such a remarkable degree, both in Roman and mediaeval times precisely *because* their stability is independent of scale and hence *because* building (i. e. testing) a scale model is a reliable way of predicting the behaviour of a full-size structure.

The designer of the church of San Petronio in Bologna (1390–1437), Antonio di Vicenzo, visited both Florence and Milan cathedrals while they were under construction. On his return to Bologna he commissioned a model of his proposed design made of brick and plaster, at about one-eighth scale. At around nineteen metres long and six metres high, it was large enough to walk inside. While probably not built deliberately to test the structural viability of the design, it certainly would have served this purpose. Most likely, such a model would have been assumed to demonstrate that the proposed church would be stable, just as it would give a reliable idea of what the future church would look like, inside and out. The fact that the church was built and was stable would have shown that such

a belief had been justified. From today's point of view we can use statics to argue the same point, but it would be no more convincing.

The idea that physical quantities vary in direct proportion was prevalent throughout the middle ages and well beyond the Renaissance. The notes made by Leonardo da Vinci (1452–1519) on structures (indeed, on most scientific matters) represent a review of scientific thinking, not only of his time, but for the previous centuries, for he added no significant new understanding to the sciences. Thus he correctly illustrates a number of linear relationships, such as the increasing deflection of beams with load; but he incorrectly suggests that the buckling strength of a column is inversely proportional to its length. Such errors were only corrected during and after the seventeenth century when the rigour of experimental science began to yield its first results (Figs. 3, 4).

Leonardo mentions several times the benefits of undertaking tests on models as a way of understanding the real world, but leaves little conclusive evidence that he did so. In his work as a military engineering he must have built models of some of the weapons he designed, and it seems likely he built and tested a model of a river or canal to determine the rate of flow he should expect in the completed works.

From the year 1585 we have conclusive evidence of models being used to investigate an engineering design problem, albeit not for a building. This was the year that plans were being made to move the Egyptian obelisk in the Vatican in Rome to a new location a few hundred metres away. Models were made of several proposals for lowering and raising the 300-tonne obelisk to help choose the most suitable method, the one proposed by Domenico Fontana (1543–1607). Apart from helping convince people his method would work, the model would also have served to plan the lowering and raising operations down to the last detail and to train the gang leaders responsible for the thousands of men operating the hundreds of winches needed to lift and manoeuvre the object. While the model would not have provided any useful quantitative data, this did not mean that

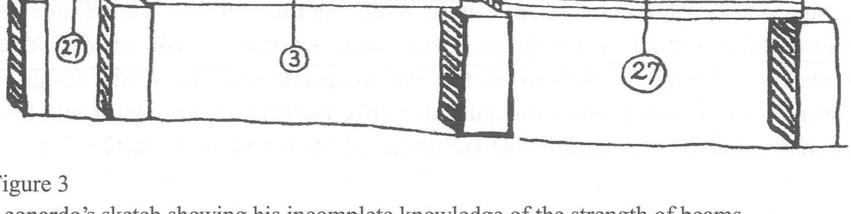

Figure 3
Leonardo's sketch showing his incomplete knowledge of the strength of beams

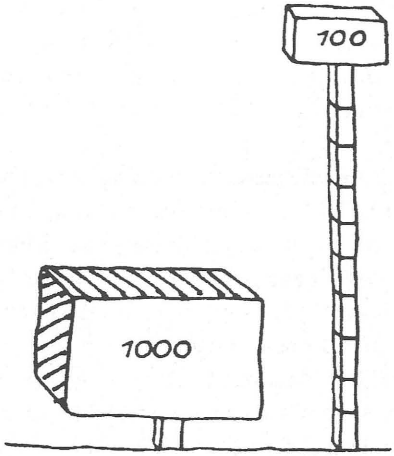

Figure 4
Leonardo's sketch showing his incomplete knowledge of the bucking strength of columns

Fontana did not consider the magnitude of the forces involved. He calculated the weight of the obelisk as 963,537 and 35/48 Roman pounds (about 309 tonnes) —a supposed one-in-ten million accuracy intended, no doubt, to impress his patron rather than the organiser of the lifting operation. From the weight and the capacity of each rope and winch, he calculated the number of winches he would need (Fig. 5).

Perhaps the earliest example of building scale models specifically to investigate the stability of a masonry structure was the work of Danizy who used plaster models of arches to study various collapse mechanisms in the 1720s. However, he did not use these model tests to collect quantitative data, nor to aid the design of a full-size structure (Fig. 6).

One of the earliest references to the sue of a model test to predict the strength of an unbuilt structures occurs in a book by the German bridge engineer Walter. In 1766 he described a one-twentieth scale model he made of a 250-foot span timber bridge which he loaded with "10 hundredweight and 10 strong men" (a total of about 25 hundredweight or 1.25 tons). He used this test to argue that the full-scale bridge would carry at least 25 tons. Although this indicates he was unaware that a linear scaling factor was not appropriate, other engineers of the day were aware of the non-linear scaling effects, and so it is likely that the capacity of 25 tons already included an adequate safety factor (Addis 1970, 156) (Fig. 7).

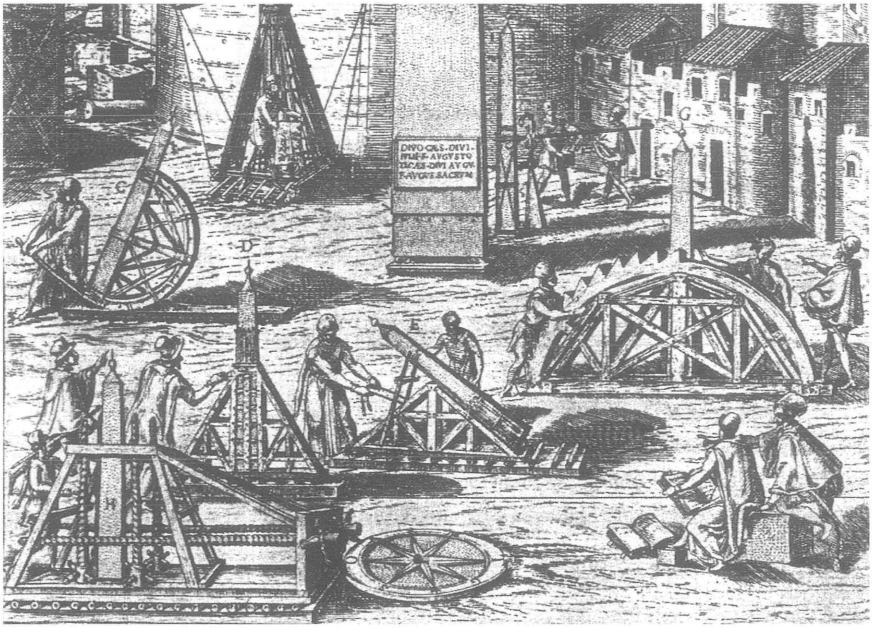

Figure 5
Models were used to test the merits of different methods proposed for moving the obelisk in St Peter's Square in Rome

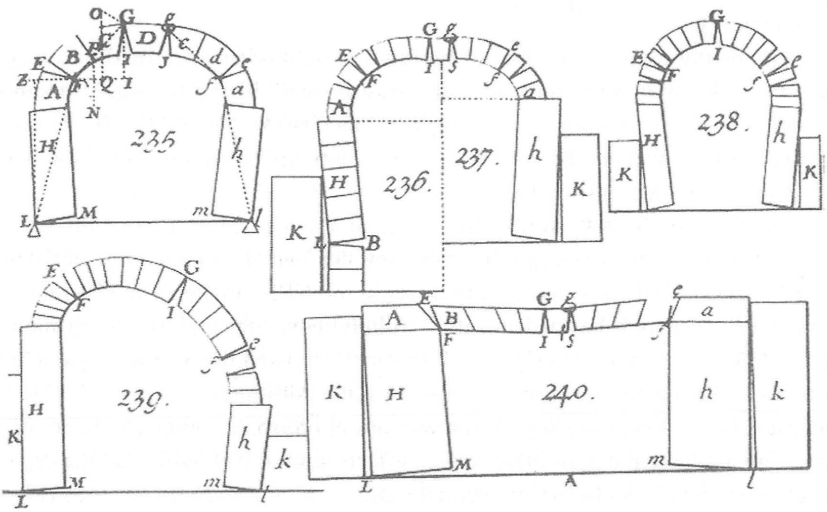

Figure 6
Model tests demonstrating collapse mechanisms of masonry arches by Danizy

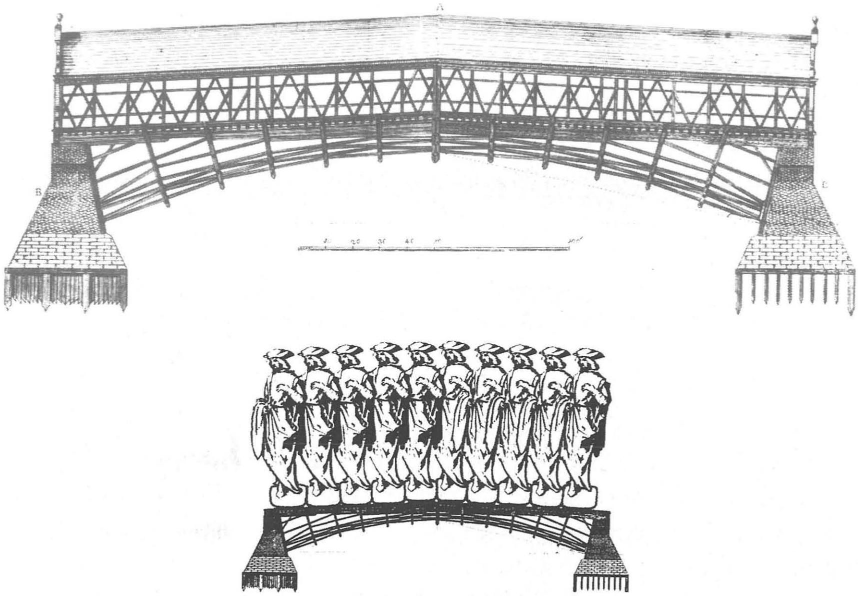

Figure 7
Load test of a 1:20 scale model of a timber bridge (Image: Walter, *Brückenbau*, 1766)

Form-finding for tension structures

Just as the stability of a compression structure is independent of scale, so too the form of a hanging chain or series of weights linked by a string. Both Leonardo and Simon Stevin illustrated such simple tensile structures and indicated how the form was governed by the relative sizes of the weights, not their absolute magnitude. These would, then serve as models of a full-size structure, had any such structure been conceived at that time.

It was not until the early nineteenth century that such models were used, for example by Telford in 1814, in the design of early suspension bridges (Smith 1977). While these models seem mainly to have been used to establish the geometry of the catenary and how it varied with the loads hanging from the wire while maintaining the tension at the ends of the catenary constant. By this time, the shape of the catenary and the magnitude of forces in it could be satisfactorily calculated using statics, so the main function of the model tests was to verify the validity of the mathematical model (Fig. 8).

In the mid-twentieth century Frei Otto and his colleagues in Stuttgart developed the use of models for form-finding in the design of cable-net structures

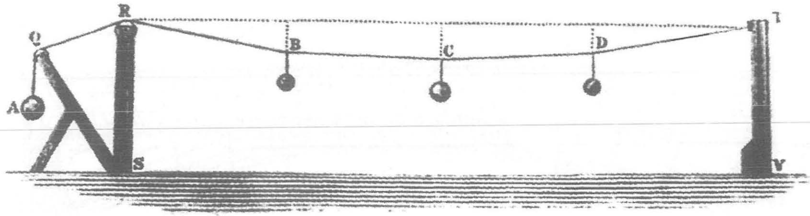

Figure 8
Telford used a model to check how the force in the suspension chain was related to its shape for his proposed suspension Bridge across the Menai Straits, Wales

and structures made from woven polyester fabrics. Since gravity loads played a minor part in establishing the form of the tension structures, the models were themselves made of membranes —either of soap bubbles, up to a metre across, which have a constant surface tension, elastic sheet whose surface tension depends on the strain, or nets (usually women's stockings) whose surface tension arises partly from the elastic extension of fibres and partly from shear deformations of the net (squares to rhombuses) (Bach, Burkhardt and Otto 1988; Addis 1994) (Fig. 9).

Form-finding for compression structures

Robert Hooke noted in 1676 that the stable form of an arch was the same, inverted, as a hanging chain comprising the same weights. He used this knowledge to help

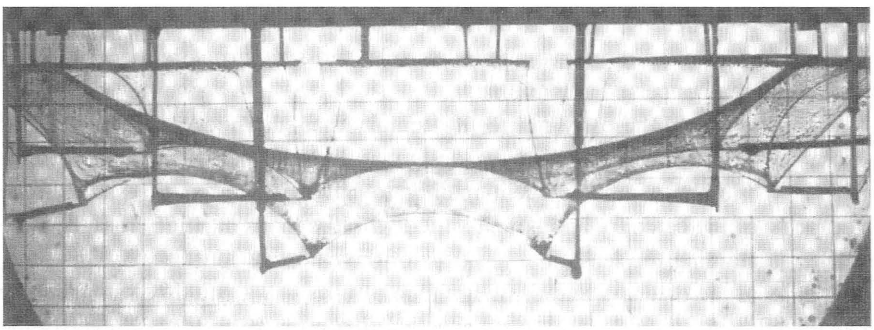

Figure 9
Frei Otto used bubble models up to a metre across to establish the forms of membrane structures (Image: Institute of Lightweight Structures, Stuttgart)

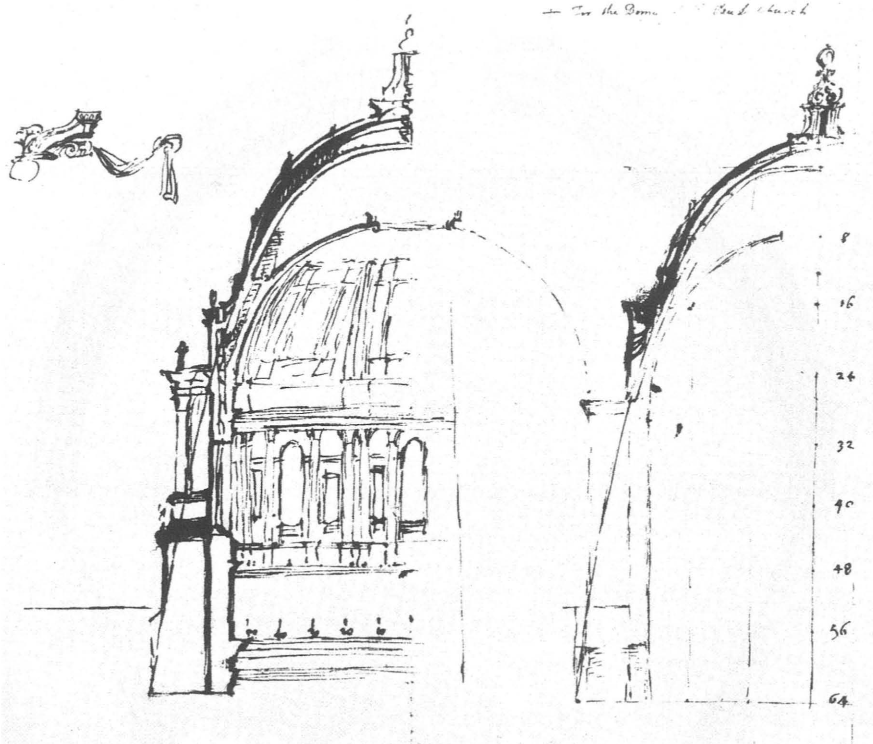

Figure 10
Wren showed the line of a hanging chain in a proposed section of St Paul's Cathedral

Christopher Wren establish a suitable form for his dome over St Pauls Cathedral in the 1680s, though we do not know if a physical model was made (Fig. 10).

A century later, Poleni used the same idea to study the stability of the dome of St Peter's Cathedral in Rome and to reassure the Vatican that the dome was in no danger of collapse. Poleni made a model with a series of weights representing the masses of the stones in the dome (Fig. 11).

At the end of the nineteenth century Gaudi used three dimensional hanging models to establish the forms of a number of arches and vaults for his buildings, most notable the chapel a Guell and the cathedral of the Sagrada Familia in Barcelona (Tomlow 1993) (Fig. 12).

In the twentieth century the Swiss engineer Heinz Isler used hanging models to establish the form of a number of reinforced concrete shell roofs he

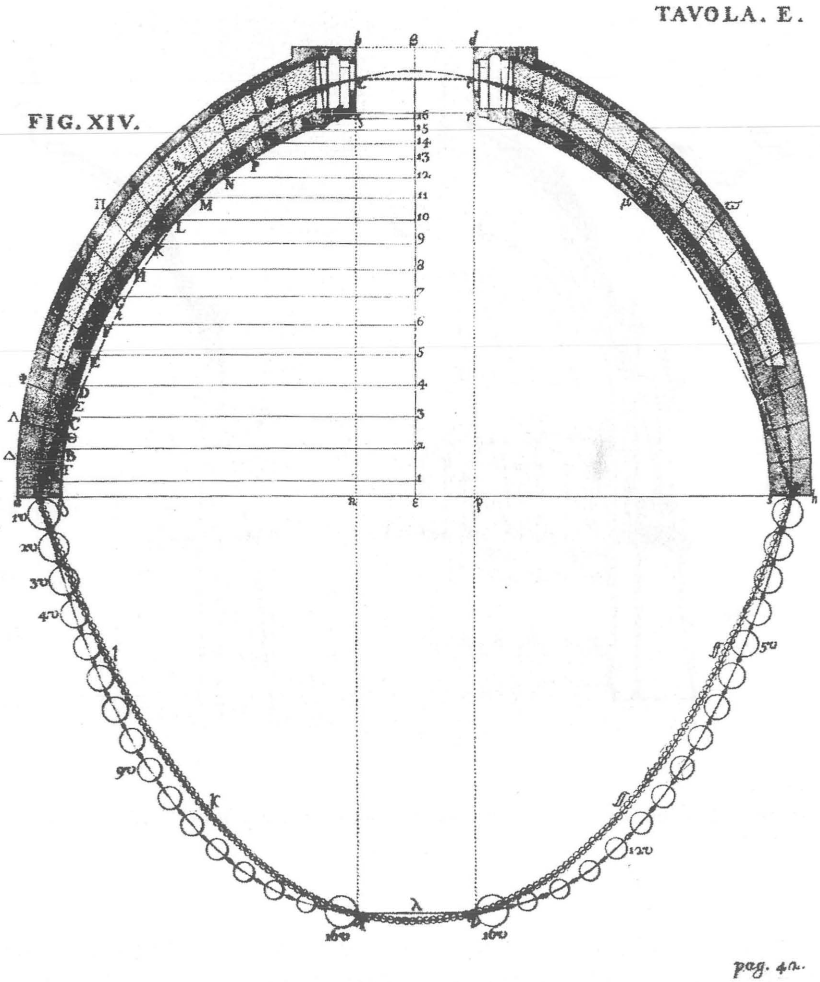

Figure 11
Poleni's drawing of a hanging chain model of St Peters indicates he actually made it

designed. He used fabrics to generate the form and made the form rigid to allow it to be accurately measured in one of two ways. He either soaked the material in plaster and allowed it to harden or soaked it in water and literally freezing the shape by leaving it outside overnight in the Swiss winter. By this means Isler generated the form whose structural behaviour he could then investigate using a statical model to ensure that suitable reinforcement was

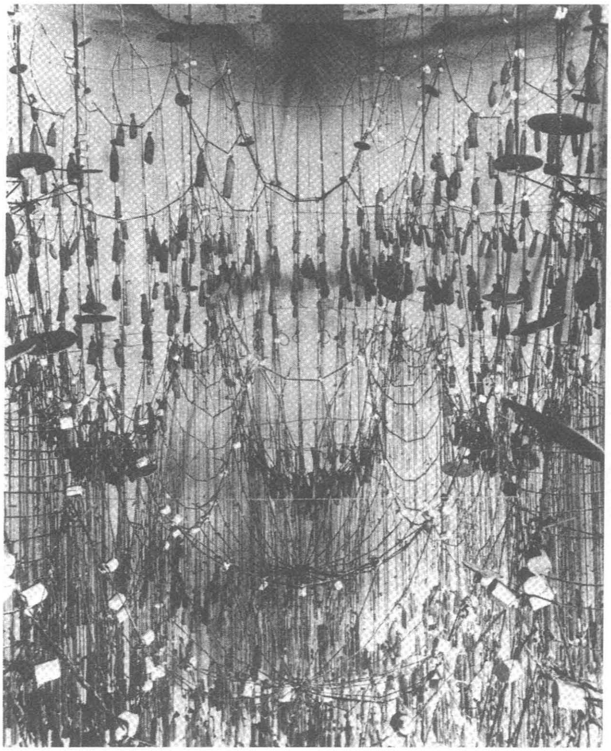

Figure 12
Gaudi used sand-filled bags and wires to create a three-dimensional shape (inverted) for the vault of his chapel at Parque Güell

included to achieve the necessary strength, stiffness and resistance to buckling (Figs. 13, 14).

Frei Otto and his colleagues also developed the use of hanging models to establish the form of a number of compression structures, most notably the timber gridshell roof for a garden festival in Mannheim in 1973. The hanging chain model provided the initial form used to create the model that was analysed by computer, at a time when computers performed iterations many millions of times more slowly than today (Figs. 15, 16).

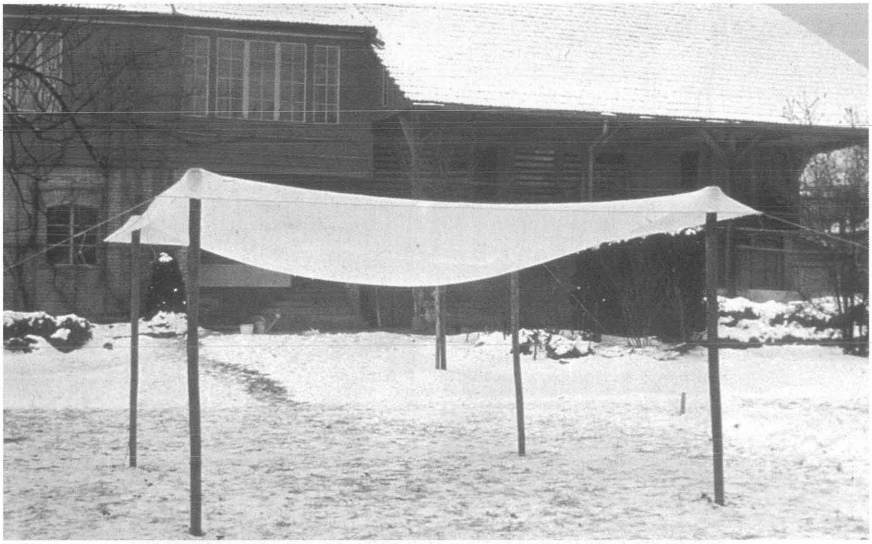

Figure 13
Isler created the form of concrete shells by freezing wet cloth (Image: Heinz Isler)

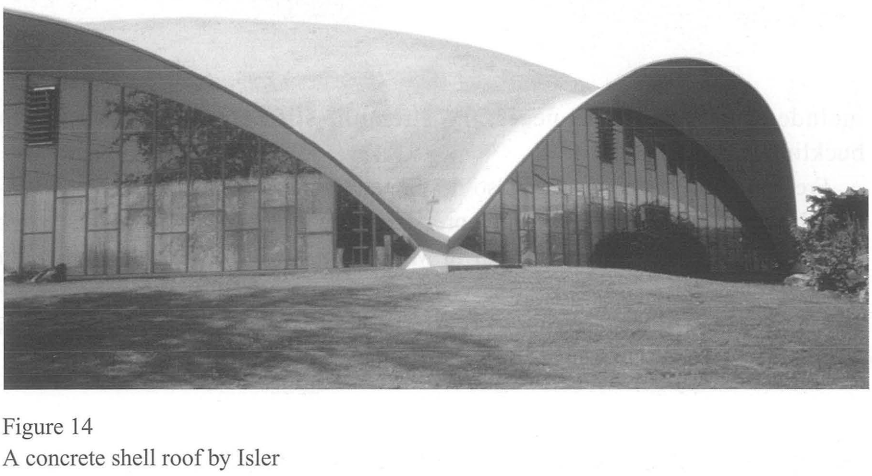

Figure 14
A concrete shell roof by Isler

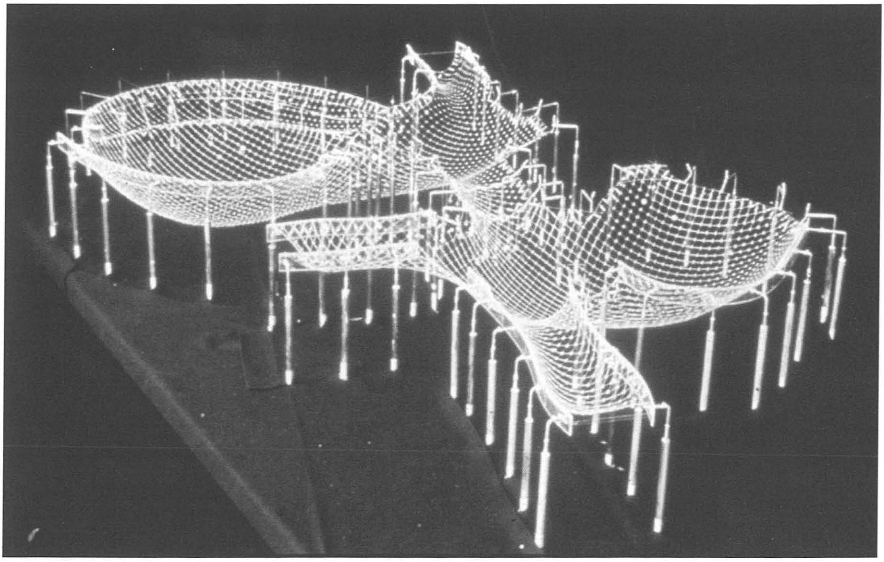

Figure 15
Hanging chain model used to establish the form of the timber gridshell for the Mannheim flower festival, 1973 (Image: Institute of Lightweight Structures, Stuttgart)

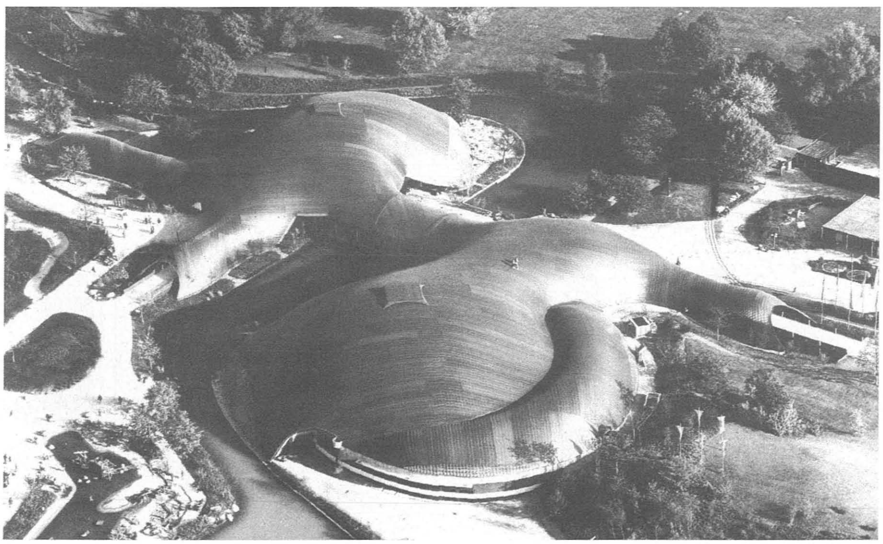

Figure 16
The timber gridshell at Mannheim covers over 5000 square metres (Image: Ove Arup)

Non-linear relationships. Scale-dependent models

The rule that physical quantities were believed to be related in simple proportion until the end of the Renaissance is proved, so to speak, by an exception from Greek military engineering from around 400BC. Engineers designed their war machines in a manner similar to that described by Vitruvius for designing temples. All the dimensions of the whole were related back to a single key dimension or *module*. In the case of a spring operated catapult, the key dimension was the diameter of the bundle of elastic cords that formed the spring (Fig. 17).

The power of the machine, and hence the weight of projectile (M minas) it could throw the distance required to damage the enemy, was governed by the diameter (D dactyls) of the bundle of elastic cords forming the spring as follows.

$$D = 1.1\ \sqrt[3]{(100M)}$$

While this is the only known example of a cubic relation appearing in engineering literature until the seventeenth century, it was an empirical design formula reflecting the outcome of much testing and experience in the field, rather than being based on a scientific understanding of the mechanics involved.

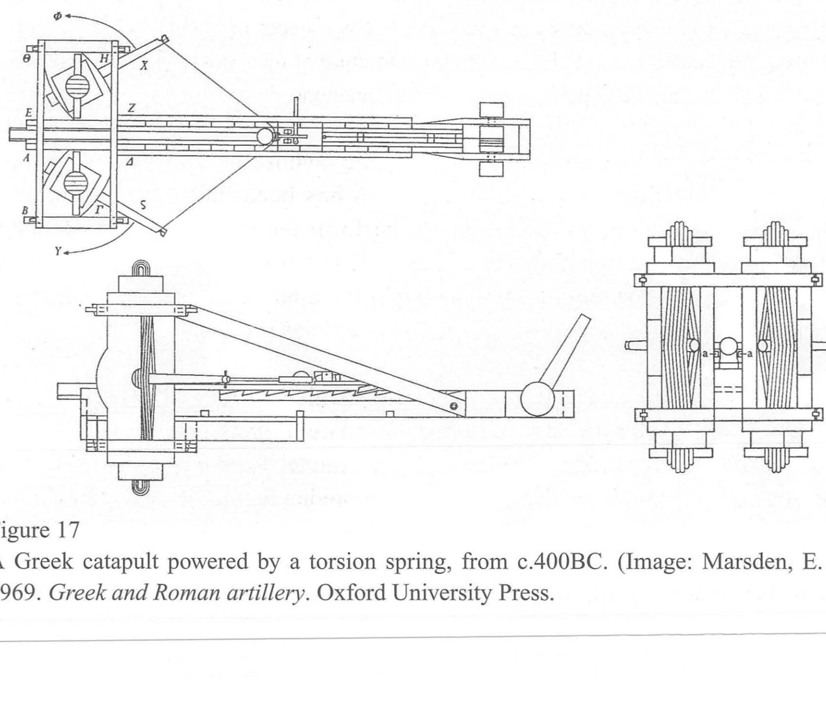

Figure 17
A Greek catapult powered by a torsion spring, from c.400BC. (Image: Marsden, E. W. 1969. *Greek and Roman artillery*. Oxford University Press.

It was around 1600 that several scientists discovered or realised that some relationships in the physical world were not linear. Simon Stevin calculated that that the distance covered by a falling body varied as the square of the time of flight —$S \alpha T^2$. In 1618 Johannes Kepler proposed his third law of planetary motion, that the durations of their orbits (T) and their distance from the sun (D) were related as $T^2 \alpha D^3$. A little later, Galileo calculated that the period of pendulum was varied with the square root of its length. In the 1650s Robert Hooke recognised that the gravitational attraction between bodies varied as the inverse square law. The way was now open to imagine non-linear relationships in any field of physics and engineering.

The first example we know of recognising the significance of scale in structures was Galileo, in 1638. He quotes a version of what we now know as the "square-cube law" —the weight of an object increases with the cube of the linear increase while area, and hence any behaviour governed by stress, varies as the square of the linear increase. He used the bones of animals as an example of the phenomenon. More generally, he recognised that scale is a crucial factor when the behaviour of a structure is being studied.

> You can plainly see the impossibility of increasing the size of structures to vast dimensions wither in art or in nature; likewise the impossibility of building ships, palaces or temples of enormous size in such a way that their oars, yards, beams, iron bolts, and, short, all their other parts will hold together. Nor can nature produce tress of extraordinary size because the branches would break down under their own weight; so also it would be impossible to build up the bony structure of men, horses or other animals so as to hold together and perform their normal functions.

To illustrate this, Galileo sketches a bone "whose natural length has been increased three times and whose thickness has been multiplied until, for a correspondingly large animal, it would perform the same function which the small bone performs or its small animal". It is not clear, however, quite what functions Galileo had in mind for the larger bone has a diameter nine times the smaller one —a cross sectional area eighty one times greater.

His lack of practical experiments places Galileo firmly in the pre-scientific era. Within a few decades the world had moved on. Edmé Marriotte (1620–1684) and Robert Hooke, (1635–1703) conducted model tests on beams to test the ability of mathematical models of bending to predict their strength. They did not, however, use their results for the design of beams —but then there was hardly any need for new data. Carpenters and builders were familiar enough with such a common structural form.

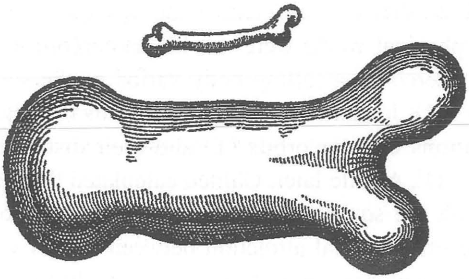

Figure 18
Galileo illustrated that an animal three times as large would need bones more than three times thicker (though not nine times thicker as he showed)

The eighteenth century saw a number of scientists, notably Musschenbroek (1692–1761) and Buffon (1707–1788), publishing the results of their tests on structural models intended especially for the designers and builders of structures. They published tow types of data —the strengths of materials including many different types of timber and iron, and the strengths of beams and columns made of these materials. The material properties were independent of scale while the data about structures had to be scaled up to the size of the real structure. The process of scaling-up was based on the mathematical models of bending developed by Marriotte and others, but incorporated empirical constants to cater for the inaccuracies in the early theories of bending. Although their work came under the criticism of some observers who still doubted the validity of scaling up results from model tests, the data published by Musschenbroek and Buffon was widely used by engineers.

By the mid-nineteenth century model testing had become a central tool in the work of engineering scientists in their search for better models of structural behaviour. Generally, however, scientists were able to simplify the experiments by excluding factors that might complicate the relationship between a mathematical model and a physical model. The idea of the pin-jointed truss was conceived with this end in mind. Engineers dealing with the real world still regarded mathematical models of their structures with suspicion as they could not reflect the real world of rigid or partially-rigid connections, or the complexities of girders made by riveting many individual pieces together. Testing a physical model gave engineers some comfort that real-world conditions could be better reflected than in a mathematical model, however their limitations were recognised (Smith 1976–77). When William Barlow undertook model tests to establish the effects of stiffening the deck on the deflections of a suspension bridge in the 1850s, he admitted they

were largely of a qualitative nature since it was not possible to reproduce at reduced scale all the features of a real suspension bridge (Smith 1977).

With little doubt, the first large-scale project for which a programme of model tests formed an essential part of the design process was the Britannia Bridge, the huge, wrought-iron tubular bridge designed and erected in 1847–49 in North Wales under the direction of Robert Stephenson. The tests were undertaken by William Fairbairn and Eaton Hodgkinson at Fairbairn's shipyard in London (Smith 1992). The proposed bridge was to be in the form of a tube through which trains would run, with four spans resting on five supports, the largest of which was about 150 metres. This was of unprecedented size and tests were undertaken to establish the best cross-section (oval or rectangular) and the best means of preventing the thin plates of iron from buckling under compression. The model of one of the spans, built at one-sixth scale, was about 27 metres long —itself quite a sizable bridge. The model tests established that the rectangular section was best and that the most economical means of resisting buckling of the compression side of the beam was to use a series of circular bundled tubes or cells rather than thick plates.

The work by Eaton Hodgkinson and Fairbairn should not be over-estimated. The conclusion that a series of bundled circular tubes would best resist buckling in the compression flange was over-ruled by the practical consideration of ease of construction and maintenance; in fact, rectangular cells were used. While the tests were successful in identifying how and where the tube would buckle, it was not at that time possible to account for scale effects in the buckling behaviour. Even for such a high-profile and large project, pressures of time prevented further tests being undertaken that might have developed the necessary understanding of buckling to assist the designers. Hence no useful quantitative data related to buckling was generated. The main quantitative outcome was the prediction of deflections of the girders and demonstration that suspension chains would not be necessary (Figs. 19, 20). In summary, then, model tests on elastic structures up to the 1860s did little to help designers with quantitative predictions. One reason was the level of understanding of structural behaviour and of how to deal with scaling effects. Another was the lack of time or money to undertake model tests on real projects. Most significant, though, was the absence of a pressing need for model testing of structures —design engineers and contactors had built up a considerable amount of experience which had defined the limits of what was wise to undertake and, generally, they kept within those limits. Generally speaking, model tests were seldom able to provide design engineers with data that would have made them worthwhile.

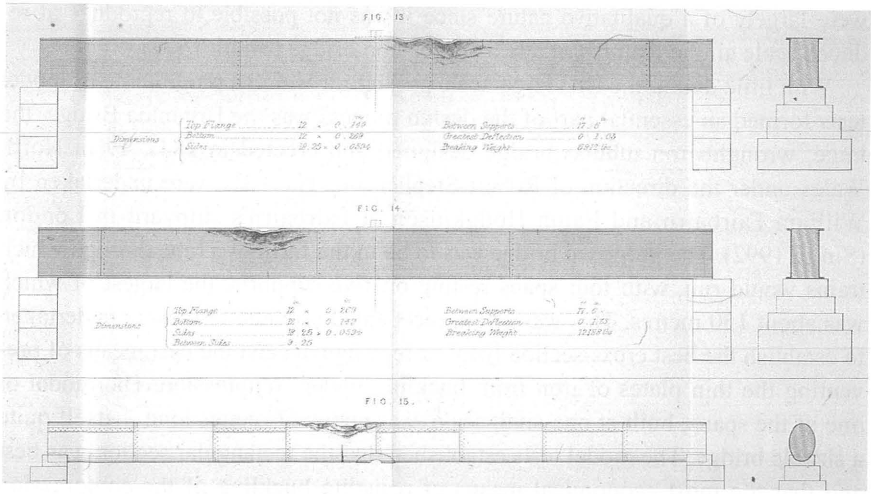

Figure 19
Results of bending tests on a 28-metre, one-sixth scale model of the Britannia Bridge to establish how best to prevent buckling in the compression zone

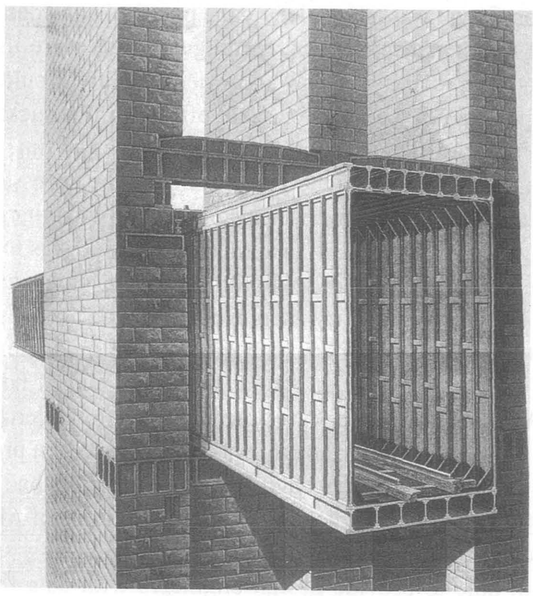

Figure 20
Britannia bridge. Cross section showing square, tubular cells to prevent buckling in the top of the beam (at mid span) and the bottom (over the supports)

Use of dimensionless constants

The breakthrough that enabled model tests to become a more useful tool to the engineer came in the 1860s and was first developed by William Froude in the field of fluid dynamics. This was the idea of a "dimensionless constant" that encapsulated the relationship between the parameters relevant to a physical phenomenon. Because the number was arranged to have no dimensions or mass, length or time, its value would be independent of scale. Hence the results of a model test could be scaled up while taking into account the different ways in which variables such as force, speed and length would be affected by scale.

Froude had worked with Isambard Brunel on the design of the Great Eastern, the largest ship in the world in 1853. He realised that friction was not the only resistance to the motion of a ship: the surface wave created by a ship also impeded its movement. He created a dimensionless constant that related the variables, a constant that would become known as the Froude Number:

$$V / \sqrt{(gL)}$$

Where V is the velocity of the ship, and L its length.

This insight led him to believe, against the opinions of all others, that models of ships' hulls might be used to predict the resistance to movement of full-size ships. He was able to conclude that the wave resistance of the model and a full-size ship would be proportional if the Froude Number for the model and proto-

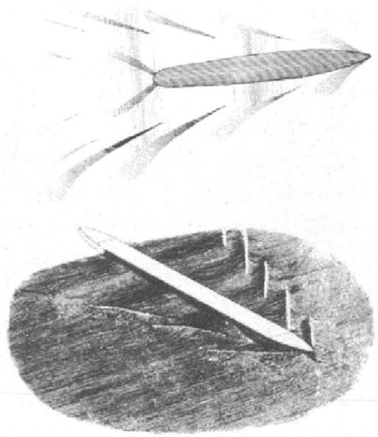

Figure 21
Froude Drawing of waves from model ship

for the firm Dykerhof and Widman which pioneered the construction of shells. Dischinger used model tests both to develop his qualitative understanding of the behaviour of shells and to test the theoretical predictions based on his and Finsterwalder's work (Fig. 24).

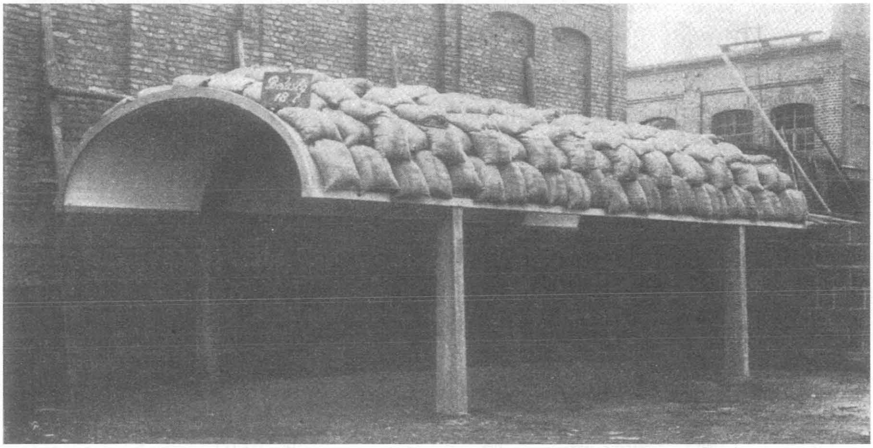

Figure 24
Model test of concrete shell roof for exhibition, 1926

Eduardo Torroja used model testing developing the design of his Fronton Recoletos in 1935 (Torroja 1967) and Luigi Nervi used model testing for the aircraft hanger at Orvieto in 1936 (Nervi 1966) (Fig. 25).

The use of models in structural engineering proved an attractive avenue for research from the 1950s, especially in the field of concrete shells in University laboratories and research institutes such as the Cement and Concrete Association in Britain. The establishing of these centres of expertise meant they were available for use by designers of shells and this became common for large concrete shells in the 1960s and 1970s (ACI 1970, Hossdorf 1974, Preece & Davies 1964). In Australia, largely thanks to the efforts of H. J. Cowan, the design code for concrete structures was amended specifically to include model testing as an alternative to theoretical analysis as a means of justifying a proposed design (Cowan et al. 1968) (Fig. 26).

Model-testing for the behaviour of buildings in earthquakes

The first systematic research into earthquake-resistant timber housing began in 1929 both at the Tokyo Earthquake Research Institute and by Professor Lydik Jacobsen at the University of Stanford. Jacobsen recognised the need for more

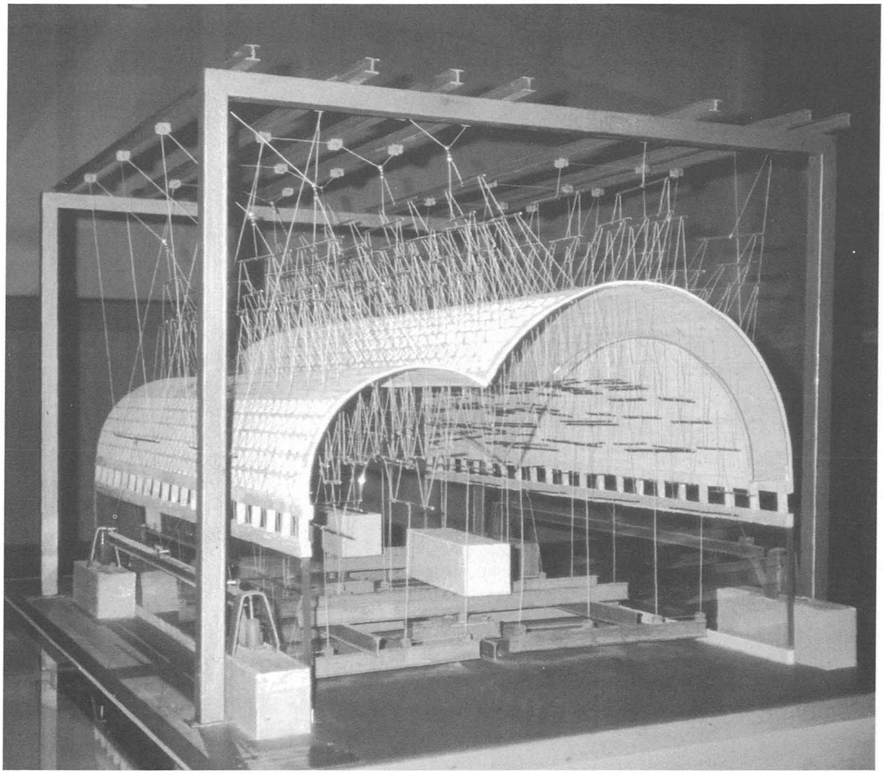

Figure 25
Model used to establish bending moments in the thin concrete shell for the roof of Eduardo Torroja's Frontón Recoletos sports stadium, 1935

sophisticated models to enable the behaviour of high-rise buildings to be studied and in 1931–33 worked with research students to undertake the first shaking-table model tests on a multi-storey building —a proposed 15-storey building for the Olympic Club in San Francisco. The model was both complex and difficult to construct. Each storey had five degrees of freedom to move —two horizontal, one vertical and two rotational, about horizontal axes. It was a mass of aluminium and steel plates and tubes, steel springs and steel ball bearings which allowed each floor to roll. The analysis raised the understanding of the effects of earthquakes on high-rise buildings to a level that allowed structural engineers to consider loads due to earthquakes when designing their buildings, and hence provide structures that would safely resist these loads (Fig. 27).

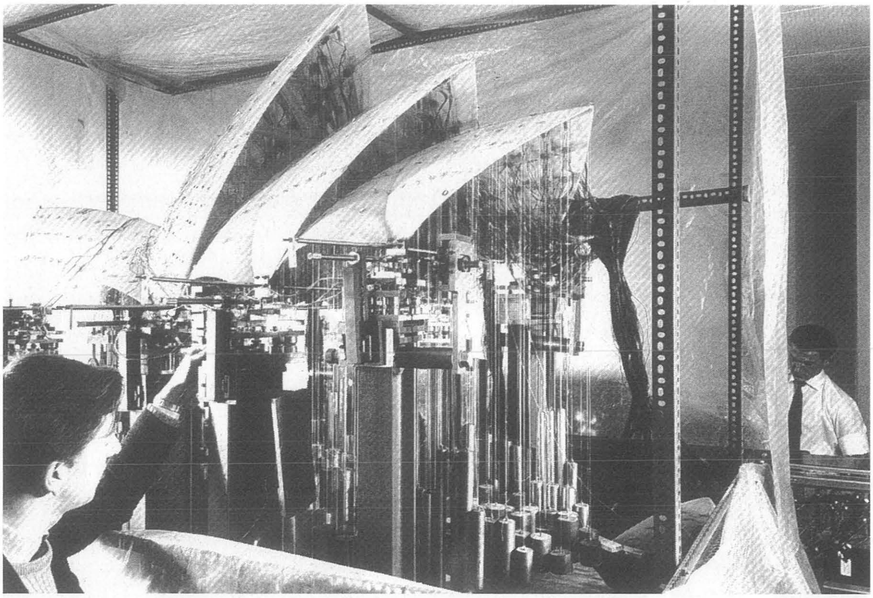

Figure 26
Model test to establish bending moments in a shell-roof option for Sydney Opera House (not as built) (Image: Henk Snoek)

Experimental stress analysis

Essential to the use of structural models to predict the behaviour of full-size structures has been the ability to measure the required variables —especially deformation. Although structural engineers are mainly interested in forces and stresses, these are impossible to measure directly. Also, it is only possible to measure deformation or strain at the surface of the model, whereas the engineer is usually interested in strains and stresses *inside* the material.

Applied loads and internal forces in bars or cables are relatively easy to measure, either by applying them directly using known weights, or by using a calibrated spring. The deformation of a model, however, is more difficult to measure. Simple extension or deflection can be measured using scales and great accuracy could be achieved using a micrometer screw gauge or clock gauge which can magnify movements. The accuracy of scales was vastly improved by the invention of the Vernier Scale, devised in 1631 by the French military engineer, Pierre Vernier (1584–1638), for use on his surveying instruments. By the end of the nineteenth century, an accuracy of 1 in 10,000 was not difficult to achieve on

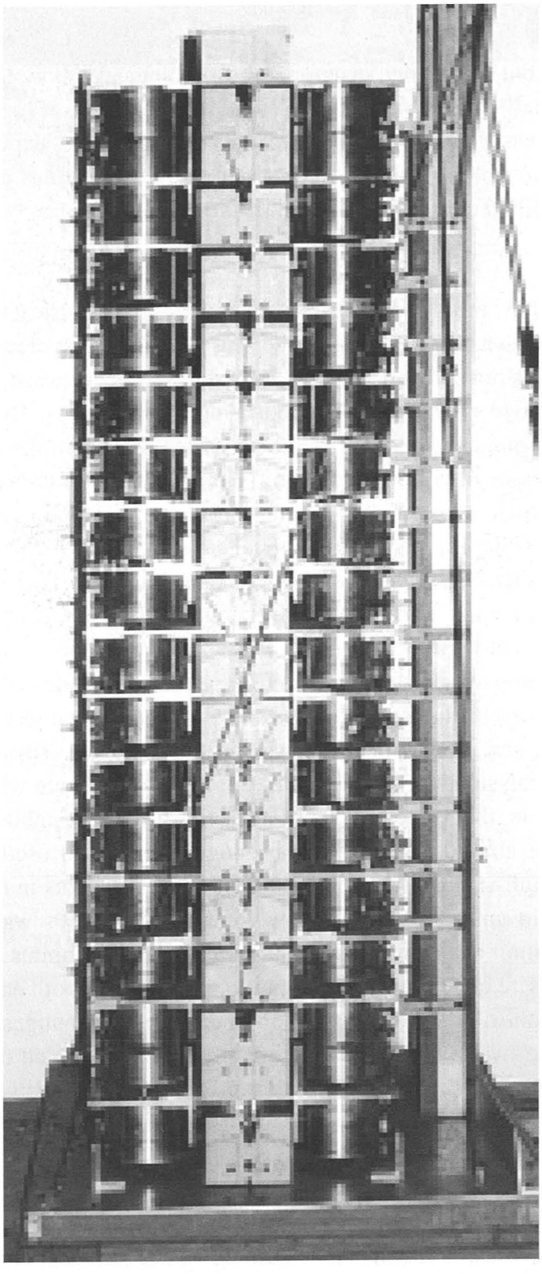

Figure 27
Model tested on a “shaking table” to establish forces and bending moments arising from accelerations caused by earthquake loading. (Image: John Blume Center for Earthquake Research, Stanford University)

large structures but for model structures the problem was how to get sufficient accuracy in a small device that could be fitted to a model.

Only in the twentieth century did the activity known as “experimental stress analysis” begin to flourish and it did so along two distinct lines of development, based on the strain gauge and on photoelastic stress analysis.

Strain gauges

The first and still the most widely-used stain gauge is the electrical resistance strain gauge. This depends upon the fact that the cross-section of an electrical conductor will reduce when stretched and this, in turn, increases its electrical resistance. This change in resistance is usually measured using a Wheatstone Bridge. Electrical resistance strain gauges were first used by Charles Kearns in the early 1930s for studying the stresses in aircraft propellers. He ground flat a conventional carbon composite electrical resistor and mount it onto an insulating strip which he cemented to the blade. He made the electrical connection to the Wheatstone Bridge by means of brushes and rings similar to those used in electric motors. In the late 1930s both Arthur Ruge and Edward Simmons in the USA used arrays of fine wires to achieve the same effect. In 1952 Peter Jackson, working with the Saunders-Roe company on the Isle of Wight, devised the idea of using the new printed circuit board technology to make foil strain gauges that were much smaller, easier to mount, and more reliable than wire strain gauges. These still form the basis of stress analysis, both singly and in the form of a rosette with three gauges at 120° to enable the principal stress directions to be established. Although developed in the aircraft industry, strain gauges were soon used for measuring stresses in both full-size and model structures and components in other industries.

Another strain gauge technology developed in the 1950s was based on the long-known relation between the pitch at which a string vibrates and the tension in the string. A wire, perhaps 200–400 mm long is fixed at both ends and its pitch measured electronically as the tension, and hence length changes due to relative movement of the two ends. Vibrating-wire strain gauges which are bulkier than electrical resistance strain gauges were used mainly for studying building elements and bridges which need to be monitored over long periods of time. They are also used in geotechnical investigation of rock movements in mines and in bridge and building foundations. Today, the outputs of strain gauges can be fed directly into computers which can calculate and display stress levels and patterns in real-time if necessary. The arrival of optical fibres and lasers during the 1970s led to the development of optical strain gauges. These have the advantage over wire strain gauges that the precise point along the length of an optical fibre at which it is stretched can be measured. This technology can be used to measure strains in the decks of bridges.

For all its modern sophistication, the strain gauge still only measures the deformation at the surface of a model. The internal forces and stresses need to be calculated using a theoretical model of the internal behaviour of the component. The only technique by which the *internal* deformations of a structural component can be measured directly is using photoelastic stress analysis.

Photoelastic stress analysis

Photoelastic stress analysis depends upon the property of certain transparent materials to affect polarised light as it passes through, in proportion to the stresses inside the material. This phenomenon, called birefringence or double refraction, was first observed in glass in the 1815 by the Scottish physicist David Brewster (1781–1868) and given comprehensive theoretical treatment by Franz Neumann (1798–1895). Certain transparent materials display an anisotropy, called birefringence, to light. When the incident light is polarised, the birefringence has the effect of rotating the plane of polarisation of the light. The degree of anisotropy and, hence, rotation, depends on the stresses in the material or, rather, the differences between the principal stresses in the three orthogonal directions, i. e. $(\sigma_1-\sigma_2)$, $(\sigma_1-\sigma_3)$ and $(\sigma_2-\sigma_3)$. If a two-dimensional test piece is used, one principal stress is zero and the magnitude and direction of the remaining two principal stresses can be established. The only point at which the photoelastic fringes give a direct indication of stress is at the model boundary where a second principal stress (perpendicular to the surface) is also zero. The varying degree of rotation of the incident polarised light is viewed in a polariscope which displays highly colourful interference fringes. For quantitative work monochromatic light (usually sodium) is used to produce sharper fringes (Figs. 28, 29).

The early development of photoelastic stress analysis was undertaken in the early twentieth century using test models made of glass which, while readily available, is not highly birefringent; it is also difficult to cut to shape and vulnerable to fracture. The technique became more accessible with the development of transparent epoxy resins such as Araldite in the 1940s. The technique reached its peak in the 1960s in the aerospace industry, for example at Rolls-Royce in England, where two- and three-dimensional models of aeroengine components were analysed in order to find ways of reducing stress concentrations that are usually the points at which potentially catastrophic fatigue cracks originate. Though not widely used by the designers of building structures, on account of its cost, the technique has been used in the design of certain highly-loaded components and structures such as dams with very thick masses of stressed concrete. While photoelasticity has now been replaced by the use of finite-element models, it is no exaggeration to say that developing the finite element technique rested entirely on the understanding of internal stresses that photoelastic models had provided over the previous half century.

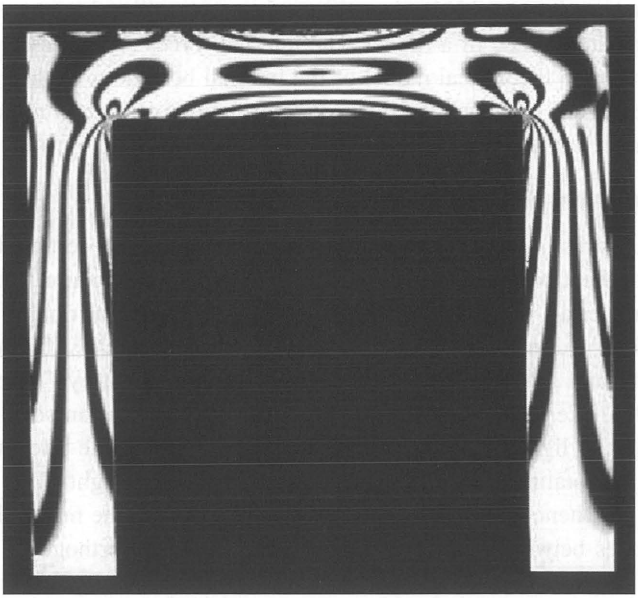

Figure 28
Stressed Araldite model illuminated by polarised light showing photoelastic fringes (isoclinic lines of constant $(\sigma_1 - \sigma_3)$)

The use of physical models today

Physical models of structures are often expensive to make and to test. Computers have enabled us to create an almost unlimited variety of models of structures and they have enabled the sharing of data between many different models. In addition to geometric and statical models of structures and elastic models of materials, we now have non-elastic structural models, models of complex and fluctuating loads, as well as models of the dynamic behaviour of structures, fire resistance and acoustic and environmental performance. Nevertheless sometimes these computer models are very costly to set up and there are still occasions when building and testing a physical model can be a cheaper and quicker alternative to using a computer model (Addis 2001).

The future of physical modelling is not without controversy. Some suggest it is no longer necessary now that computer models are so sophisticated and reliable. Others, however, still consider that physical models give an understanding of structural behaviour that no computer program can provide. In this regard,

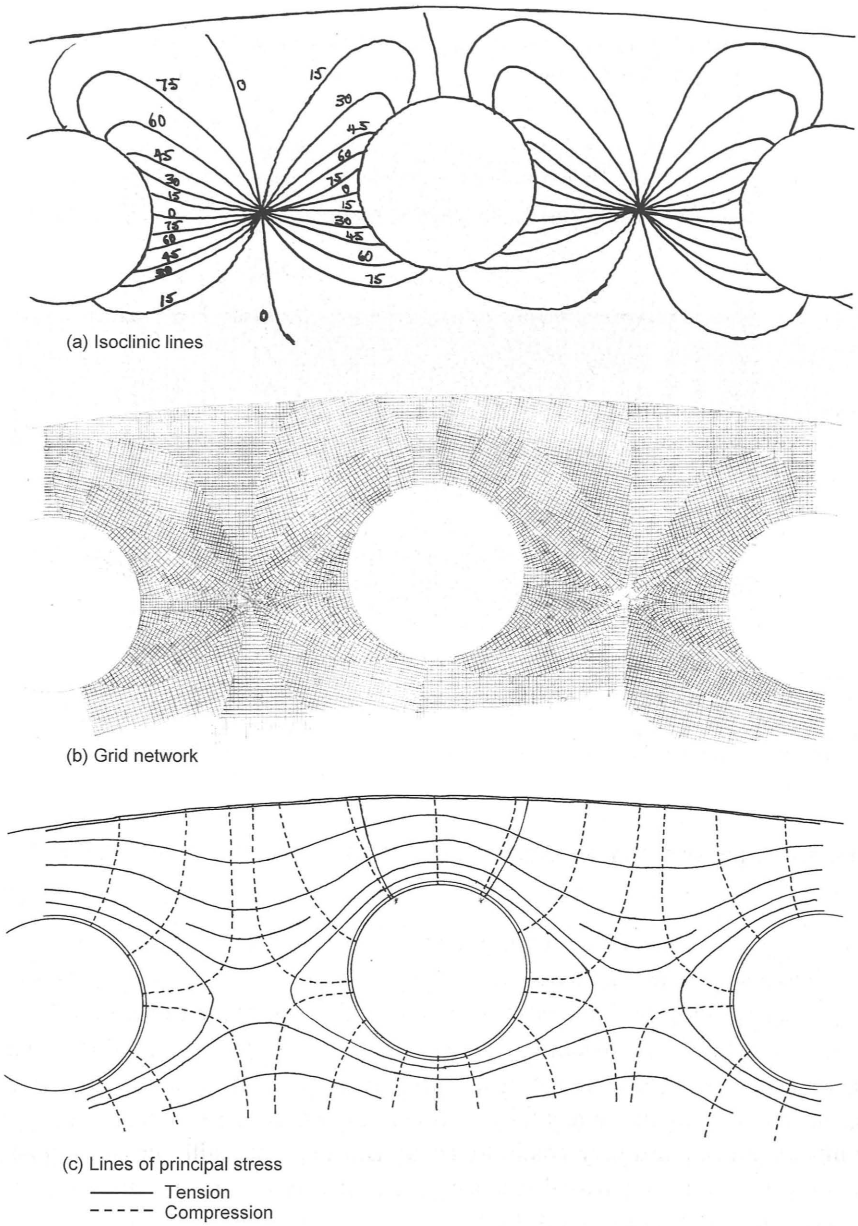

Figure 29
The manual process of constructing isostatic lines of internal tension and compression stresses from isoclinic lines

there are interesting, and perhaps significant, differences between cultures and between the education of different professions. Japanese engineers working on the design of various unusual structures over the last two decades have made much use of physical models —when a computer model was judged to be too difficult to create, as confirmation of the results of computer analysis, or to test the stability of a structure during an unprecedented erection procedure (Saitoh 2004) (Fig. 30).

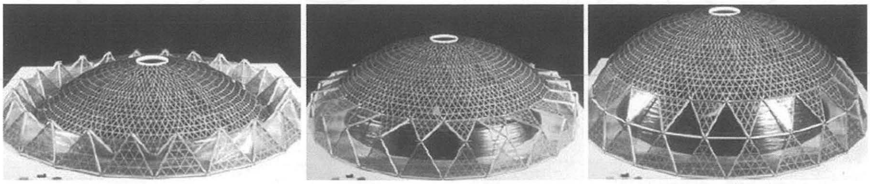

Figure 30
Model used to test the erection procedure for an articulated dome, and to verify stability in all stages (Image: Saitoh)

Building and testing a physical model can reveal a mode of structural behaviour that was not previously known. This can never be the case for a mathematical or computer model which can only be constructed according to known patterns of behaviour.

Concerns are often raised by experienced engineers that it is now perhaps too easy to build virtual models of structures and lose track of the relationship between the behaviour of the computer models and the behaviour of the real structures, materials and loads that they are believed to be modelling. Building and testing a physical model remains the only way of getting a real feel for how a structure behaves, and developing the essential link between the abstract and the real world of structures.

It is (or should be) disturbing for the structural engineering profession that building and testing models has become a rare activity in the engineers education and formation. Architecture students, on the other hand, still enjoy and benefit from model-making (Kawaguchi 2004). Unfortunately, they are perhaps attracted to the more exotic structural forms rather than day-to-day structural frames and roof trusses or the arches, vaults and domes upon which the survival of so many mediaeval and Renaissance masterpieces depend (Fig. 31).

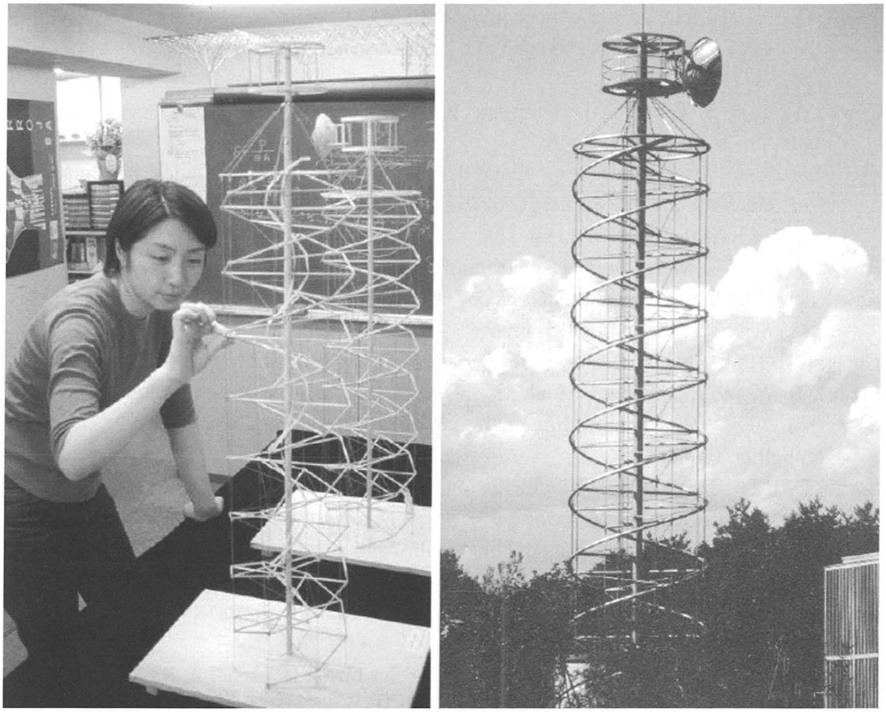

Figure 31
A 'touch and feel' model used to gain *understanding* of the stiffness and behaviour of a structure under load (Image: Kawaguchi).

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Vaults in the air: *Signor Fabritio*'s English theory

Antonio Becchi

The lies of the wise one

In long ago 1972, when some of us were still playing with Legos, Jacques Heyman sent to press a text which was to become, in a short while, a reference for all scholars of the history of structural mechanics: *Coulomb's Memoir on Statics* (Heyman 1972). This book marked a turning point, and filled in with elegance and discretion the gaps left by the articles on the history of civil engineering published by Stanley B. Hamilton beginning in the 1930s in the *Transactions* of the Newcomen Society and the great surveys of William Barclay Parsons (Parsons 1939) and Hans Straub (Straub 1949). The modesty that pervades the Preface might have fooled the unprepared reader, and the innocent lies of the author might have induced hasty judgements, but that wasn't the case. In fact, few believed what Heyman wished them to believe, that (the first lie) "although some of the older work appears to be newly rediscovered here, this whole book leans very heavily on the findings of previous commentators". It is obvious that all authors base their work on the writings of those who came before, but Heyman's study of the *Essay* of Coulomb was not just another stop on the water wagon's route; it indicated the very wellspring, up to then but little investigated. It laid down the foundations for a new mint of historiography, distancing itself, without saying so explicitly, from two current forms of historiography: one dedicated to sweeping syntheses, and the other which dedicated a few pages (not more) in journal articles to a single moment in history without examining its context in depth, barely touching on ideas and concepts.

In achieving this historiographic turning point, straddling the fence between mechanics and engineering, Heyman had a sure and recognised model, whom he himself acknowledges in the Preface: Clifford Ambrose Truesdell (Truesdell 1960). Here again, however, his appreciation carried him too far (the second lie): "The more recent technical histories are quoted in the text and listed in the references; of these, Truesdell's account of flexible bodies is definitive". The work of Truesdell, extraordinary as it is, was not definitive at the time, and is even less so today. That it is not definitive is thanks to the work of Heyman himself, and of those who, following his example, have begun once again to search where it once seemed that there was no longer anything to search for.

The essay that follows presents one example among many that show how the history "between mechanics and architecture" is still largely unexplored territory. Heyman showed us this when no one wanted to believe it, when the great historical surveys seemed to have taken away the desire to delve into analysis. These pages are dedicated to him, with affection and profound recognition.

Castles in the air

When, in 1624, Sir Henry Wotton dedicated himself to writing the *Elements of Architecture* (Wotton 1924) he had just returned from Italy, where he had been King James I's ambassador to the Venetian Republic. As a lad, he had evidenced an interest in architecture, but this might have been enumerated as one among many cultural inclinations of a talented young man destined for a career in diplomacy. His contact with Italy, where he had come often beginning in 1592, but above all the long and repeated sojourns in Venice, reinforced this passion. No one living in Palazzo Gussoni, in Venice and spending summers on the shores of the Brenta would have been immune to the seduction of the art and architecture. Nothing, however, would have led to believe that Wotton would have become the author of a treatise, albeit an anomalous one, on architecture.

In a letter addressed to Dudley Carleton, dated[1] 16 January 1624, Lord Chamberlain maliciously described the activities of the former ambassador: "I heare [Wotton] is now retiring to some corner in the countrie to finish a worke he is setting out of the mathematikes or perhaps building of castles in the ayre" (Smith 1907, 1, 194). Not even the author was clear as to his ideas on the contents of the work that he was fast and furiously drafting (it was, as he wrote, "printed sheet by sheet as fast as it was born, and it was born as soon as it was conceived").[2] His objective, however, which required much diligence, was clear: to earn him the place as Provost of Eton College, after he had renounced diplomacy. At the time Wotton was 56 years old. Paolo Sarpi, a leading light of Venetian culture and trusted interlocutor of "Italianate Englishmen"[3] had died shortly before. The

world seemed changed forever, and facts were not long in proving this: an epoch was ended, in Venice as in England, and in 1625 would begin the restless rule of Charles I. The bucolic retirement to Eton, where Wotton would be Provost from 1624 until the end of his life in 1639 represented the ideal counterpoint to the whirlwind (and splendours) of diplomatic life.

Against this background, strongly influenced by immediate contingencies, Wotton's *Elements* appear original and surprising. Even though the author himself underlines in the title that these "elements" are "collected by Henry Wotton Knight, from the best Authors and Examples", the book's structure does not align itself with current models; if anything, it proposes a new model. The reflections on the theory of vaults is ample demonstration of this. In a few pages Wotton presents the first theorems ever published on the theory of arches, and which derive, word for word, from Bernardino Baldi's comment on *Quaestio* XVI[4] (Baldi 1621, 95–114). The term "theorem", insistently used by Wotton, is certainly improper and excessively emphatic when compared to the content, but it is nevertheless undeniable that the text has an expositive originality all its own (Baldi never uses the term *theorema* in relation to the theory of arches). Here and in other passages, originality and copying of what came before are indissolubly mixed.

Futher, the *Elements* represent the first attempt to offer a new contribution in English to the field of architectural theory. The previous work of John Shute (Shute 1563), can in fact be enumerated as one among the many reprisals of the theory of the orders, without offering any innovations. The singularity of Wotton's approach to the theme of vaults receives confirmation from successive literature in English, which developed in the course of the seventeenth century totally independently of the great French contributions. The comparison between the descriptions provided by Wotton and those of Roger North or Christopher Wren in the few notes which have come down to us leaves no doubt as to the diverse nature and "tradition" of the mechanical interpretations that he proposes.

As in a mirror

In order to completely comprehend the coexistence of originality and plagiarism in Wotton's five theorems it is necessary to reread the original source for them, Baldi's *In mechanica Aristotelis problemata exercitationes*[5], and in particular his comment on *Quaestio* XVI. The subject appears almost banal (Baldi 1621, 95) (Fig. 1):

> One asks why it is so that the longer the beams are, the weaker they become; and when raised, why they bend more than short beams, which measuring, for example, two cubits are thin, and those measuring a hundred cubits long are very thick.[6]

EXERCITATIONES. 95

mnia cùm vera ſint, nemo, vt arbitror, dixerit, abſolutè, quod voluit Ariſtoteles, id ex rotatione velociori & partium à centro remotione, fieri.

QVAESTIO XVI.

Dubitatur, quare, quò longiora ſunt ligna, tãto imbecilliora fiant, & ſi tolluntur, inflectuntur magis: tametſi quod breue eſt ceu bicubitum fuerit, tenue, quod verò cubitorum centum craſſum?

EX ſuis principijs ſoluit Ariſtoteles. Inquit enim: An quia & vectis & onus & hypomochlium, id eſt, fulcimentum in leuando, fit ipſa ligni proceritas? Prior namq; illius pars ceu hypomochlium fit, quod verò in extremo eſt, pondus: quamobrem quanto extenſius fuerit id quod à fulcimento eſt, inflecti neceſſe eſt magis; quo enim plus à fulcimento diſtat, eo magis incuruari neceſſe eſt. Neceſſariò igitur extrema vectis eleuantur. Si igitur flexilis fuerit vectis, ipſum inflecti magis cum extollitur neceſſe eſt, quod longis accidit lignis, in breuibus autem quod vltimum eſt, quieſcenti hypomochlio deprope fit. Hæc ſubiectâ figurâ ob oculos ponimus.

Eſto longum ac flexile lignum AB, manu eleuetur in A, flectetur itaq; in B, & declinabit in C. etenim manus quæ ſuſtinet in A, fulcimenti loco ſuccedit: longitudo vero AB ponderis vices refert, atque vectis, quare quo longius abfuerit à fulcimento, id eſt, manu extremum B, eo magis flectetur; ſi autem lignum breuius fuerit, nempe terminatum in D, nequaquam flectetur, eò quòd eius extremum D minus à fulcimento quod eſt in A ſit remotum. Hæc igitur eſt mẽs Ari-

Figure 1
From (Baldi 1621, 95)

Behind this statement lurk various problems that would finally be resolved only when their varying natures were understood. Baldi proposed a new interpretation of the subject, which up until that time had always been reduced to the principle of the lever AB (Fig. 2), and underlined its polymorphous character. He is particularly close to the spirit of the problems posed by the *firmitas* of Vitruvius. When he moves from his judicious treatment of *Quaestio* XVI (the first three pages) to setting forth his most original interpretation of the problem, he appears almost bored by the traditional explanations. Or at least he appears to think that the reader must be bored by them. Here is the passage that marks the turning point (Baldi 1621, 98):

> Now, to derive from this study —which might otherwise seem futile— some profit, and so that our arguments might serve to render architects more prudent, we will appropriately apply our considerations to architecture.[7]

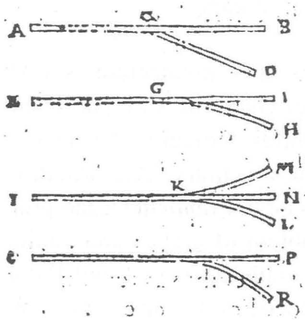
This does not deal merely with beams. It is architecture as a whole that can benefit from reflection on the treatment of *Quaestio* XVI, particularly as regards the four ancient problems, to which Baldi dedicates the next twenty pages (Fig. 3): the problem of columns (which, a century later, would be studied by Leonhard Euler as the problem of the *force des colonnes*); the problem of floor beams; the problem of trusses; the problem of arches and vaults. Obviously, twenty pages are not enough to address subjects so vast, but Baldi manages to give some general guidelines that will be decisive for future discussions.

The most marvellous thing about the treatment is the apparent, daring straying from the subject itself. It is astonishing to compare Baldi's work with the other Comments on *Mechanical Problems*, both those which Baldi certainly knew as well as those which he had probably not read at the time of writing the *Exercitationes*. While the other authors limit themselves to criticism of the Greek text, Baldi considers it a mere pretext for writing a small treatise on the art of building.

It is easy to recognize how the salient characteristics of Baldi's work appear in that of Wotton. What is being dealt with is not mere derivation, which is of course obvious in the case of the theoryof vaults, but a singular "constitutional" affinity: Baldi and Wotton are both outsiders who write about *ars aedificatoria* but who are not recognised experts in the field.[8] It would have been easy to attribute to both of them the malevolent expression of Lord Chamberlain: "builders of castles in the air". Criticisms of Baldi's comments on *Quaestio* XVI arise, at least in part, from this prejudice. For example, in a letter[9] to Pier Matteo Giordani of 1621 Muzio Oddi wrote (Oddi 1621):

96 IN MECHAN. ARIST. PROBL.

Aristotelis, cuius quidem sententiam non damnamus; quippiam tamen addimus. Dicimus autem materiam, quatenus ad hanc contemplationem spectat, in duplici esse differentia. aut enim rarefactionis & constipationis est incapax, vt in chalybe videmus, nitro, metallo, marmore, aut capax quidem, & hæc duplex: Vel enim natura nata est ad rectitudinem quandam, vt arborum flagella virgæque, aut non item, ceu stannum, plumbum, & cætera eiusmodi.



Esto primò vitreum corpus gracile, procerum, teres AB, manu capiatur in A, itaq; pondere ipsius corporis præualente ad partes B, quia in C puncto, quod circa medium est, ex parte superiori non fit rarefactio, nec in inferiori constipatio, nec interim datur penetratio corporum; fit fractio à superiori parte, & pars CB à reliqua parte AC, auulsa & separata cadit in D, succedit autem ipsa separatio rarefactioni. Porrò quod materias hasce non flexibiles diximus, sed frangibiles, non ideo negamus vel sensu docente, aliquam in ijs fieri flexionem. Si autem lignea fuerit materia, eaq; flexibilis, vt EF, si manu eleuetur in E, præualente pondere in F flectetur vbi G. ibi enim à parte superiori fit rarefactio, ab inferiori verò constipatio, & pars GF declinabit in H, quæ declinatio eò vsque procedet, quo rarefactio & constipatio competens naturæ illius materiæ, quæ flectitur ad summam intensionem deuenerint; tunc si vis maior ingruerit, frangetur omnino: si secus facta ibi resisten-

Figure 2
From (Baldi 1621, 96)

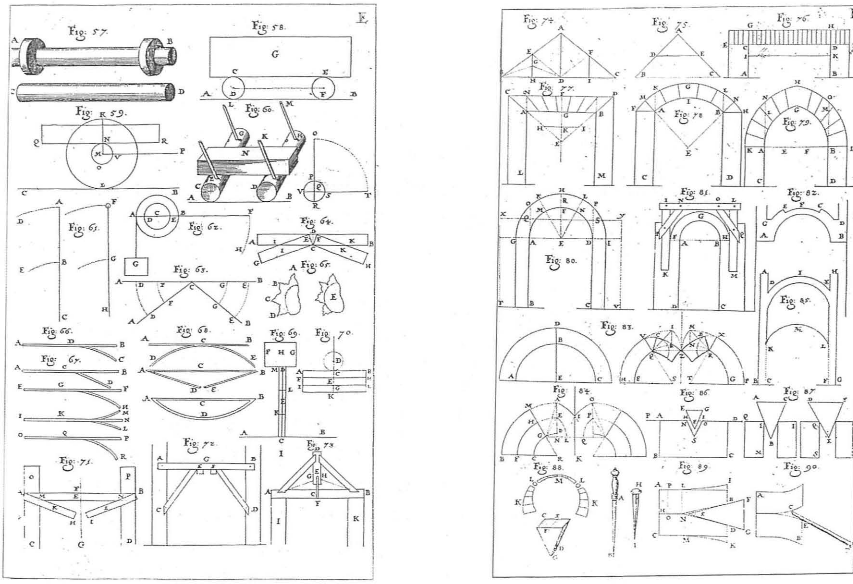


Figure 3
From (Mögling 1629, Tab. E and F). The figure 62–85 illustrate Baldi's commentary to the *Quaestio* XVI

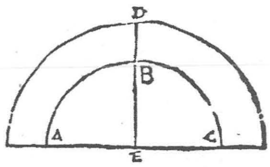
> There are passages [of the *Exercitationes*] that are certainly hardly worthy of his ingenuity, and he jumped from one thing to another, and took up certain arguments, as for example of question XVI, where he imagines the breaking of a vault in terms of a circular motion, which is not true, and he made a bagatelle seem a thing of great importance.[10]

The mechanism of the breaking of an arch was rejected across the board, though in the centuries to come it would prove to be particularly suited for describing experimental reality (Fig. 4).

This is not an auspicious beginning for a posthumous work, undoubtedly spoiled by incompetent editing. However, some years after Oddi's letter the novelties proposed in the *Exercitationes* were not lost on the sensitive eyes of Giovanni de Guevara, who is his *In Aristotelis Mechanicas Commentarij* (Guevara 1627) mentions Baldi's XVI and lets it be understood that he has it in mind. De Guevara's interest is strictly mechanical; he does not take any of Baldi's architectural considerations into account, nor does he believe to be necessary to set forward any of his own.

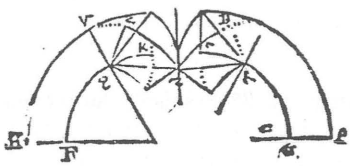
Wotton's great merit is that of having brought Baldi's comment back into the centre of attention. In the *Elements* he becomes a primary reference for the

Cæterùm admonet nos locus, vt aliquid de fornicum diſſolutionibus in medium afferamus: cauſſis enim morborum cognitis, facilius periti medici adhibere ſolent remedia.



Eſto enim ſemicircularis fornix ABC, cuius centrum E, perpendicularis verò quæ per centrum DBE, ſemicirculi ABC, diameter AEC, incumbæ vtrinq; A, C. Itaque ſi nulla fiat incumbarum repulſio, ſtabit fornix; ſi verò fiat, ruinam faciet.

Pellantur itaque ad exteriores partes, vt in ſecunda figura, H in F, & C in G,



ex qua pulſione cum maius fiat ſpatium quod integro fornice implebatur, iam diſtractis vtrinq; fornicis partibus nõ impletur, Diuiditur igitur locus maior factus in tres partes, quarum hincinde duas replent fornicis partes, tertiam verò quæ media eſt, replet inſertus, ne vacuum detur, aër, vt in figura videre eſt, in qua ſolutæ vtrinque fornicis partes HIKF, PMNG, aër autem medius ſpatium replens IKMN. Diuidantur ſinguli quadrantes FK, GN, in partes tres, quarum duæ ſint hincinde FQ, GR, & à centris, quæ ſeparatis quadrantibus facta ſunt in ST, rectæ ducantur SQV. TRX. Quoniam igitur tertiæ partes vtrinque VIKQ MNRX propria grauitate depreſſæ, nullum quo ſuſtineantur fulcimentum habent, corruent quidem. Ducantur autem rectæ QI, RM, conſtituentes cum ipſis QV, RX pares angulos VQI MRX. Itaque centris QR partes QIRM ad infe-

Figure 4
From (Baldi 1621, 112)

analysis of the "mechanical" problems of architecture. There was little else that the author could cite, even though he was familiar with a good part of the best architectural literature then available: there were rare suggestions in other sources (such as, for example, the *prova di carico* of the egg illustrated by Vincenzo Scamozzi (Scamozzi 1615, pt. 2, bk. 8: 320) in his *Idea dell'architettura universale*), and the bibliography relative to the relationships between mechanics and architecture was too meagre. Thus there were excellent pretexts for enthusiastically embracing and amplifying the scope of Baldi's proposals.

In the *Elements* Baldi assumes a role that literally reflects the spirit of the comment on *Quaestio* XVI: the one who completes the classic treatise of mathematical "proofs". His name first appears in the fourth theorem on arches (Wotton 1624, 48–50):

> Theoreme 4. If the *Materials* figured as before *Wedge-wise*, shall not be disposed leuelly, but in forme of some *Arch*, or portion of a *Circle*, pointing all to the same *Center*: In this case neither the pieces of the sayd *Arch*, can sinke downewards, through want of roome to descend[11] ["By the first Theor."] perpendicularly: Nor the *Supporters* or *Butments* (as they are tearmed) of the sayd *Arch* can suffer so much violence, as in the precedent flat Posture, for the roundnesse will alwayes make the Incumbent waight, rather to rest vpon the *Supporters*, then to shoue them; whence may be drawn an euident *Corolary*: that the safest of all *Arches* is the *Semicircular*, and of all *Vaults* the *Hemisphere*, though not absolutely exempted from some naturall weakenesse ["Which is the sole prerogatiue of perpendicular lines and right Angles"], as *Barnardino Baldi Abbot* of *Guastalla*, in his Commentary vpon *Aristotles Mechaniques*, doth very well prooue; where let me note by the way, that when any thing is *Mathematically* demonstrated weake, it is much more *Mechanically* weake: Errors euer occurring more easily in the management of *Grosse Materials*, then *Lineall Designes*.

The subject of perpendicularity, developed in the first theorem, is linked to what Baldi describes in the part dedicated to beams-columns, but which finds confirmation as well in Vitruvius's *De architectura*[12] (Vitruvius 1486, book 4, chap. 8). From the suggestion of Vitruvius Paolo Casati takes his departure in *Mechanicorum libri octo* (Casati 1684) to treat the problem of the statics of leaning towers ("cur turres inclinatae non corruant"), in which the case of the tower of Bologna is described with particular care.[13] Baldi takes up the same subject in *Quaestio* XXX.

The way in which the *Elements* mirrors the character of the *Exercitationes* can be found in other theorems, even if Baldi's name appears only in the fourth and is mentioned again some pages further on with regards to trusses (Baldi 1621,

101–104). Below is a brief outline of the subjects dealt with, with an indication of their relative correspondences:

Theorem I. The natural inclination of weights to go descend in a perpendicular direction, in accordance with the shortest path (Baldi 1621, 98 and 104–105); "*ponderosity* is a naturall inclination to the *Center* of the World, and *Nature* performeth her motions by the shortest lines" (Wotton 1624, 47).

Theorem II. The necessity of defining the *shape* and the *position* of the elements that form a jack arch (Baldi 1621, 105–106), so as to avoid the problem described in Theorem I: "Therefore to make them stand, wee must either change their *Posture*, or their *Figure*, or both" (Wotton 1624, 47).

Theorem III. The advantages that derive from the use of elements cut in the shape of a wedge. Description of the strong thrust that is generated in walls supporting jack arches (Baldi 1621, 106); "If Brickes moulded, or Stones squared *Cuncatim* [Cuneatim] (that is, *Wedge wise*, broader above then below) shall be layd in a *Row levell*, with their ends supported, as in the precedent *Theoreme*, pointing all to one *Center*; then none of the pieces betweene can sinke till the *Supporters* give way, because they want roome in that *Figuration*, to descend *perpendicularly*. But this is yet a weake piece of *Structure*, because the *Supporters* are subject to much impulsion, especially if the line be long" (Wotton 1624, 48).

Theorem IV. The way to avoid the strong thrust produced by the jack arch (Baldi 1621, 106–107): arches where the voussoirs are not "disposed levelly" (Wotton 1624, 49). From this follow the corollaries: the most secure arch is semicircular and the most secure vault is hemispherical. However, in both cases there is "some naturall weakenesse" (Wotton 1624, 49), as made evident by what was described in Theorem I.

Theorem V. Again, on the security of circular arches and hemispherical vaults, with a clarification as regards aesthetics: "So those are the gracefullest, which keeping precisely the same height, shall yet bee distended. One fourteenth part longer then the sayd entire *Diameter*; which addition of distent will conferre much to their *Beauty*, and detract but little from their *Srength* [sic]. This obseruation I finde in *Leon Batista Alberti*; But the practice how to preserue the same height, and yet distend the Armes or ends of the Arch, is in *Albert Durers Geometry*, who taught the *Italians* many an excellent Line, of great vse in this *Art*" (Wotton 1624, 50).

The conclusion of Wotton himself, placed as a seal at the end of the five theorems, is exceedingly significant: "Vpon these fiue *Theorems*, all the skill of *Arching* and *Vaulting* is grounded" (Wotton 1624, 51). Thus is made evident Wotton's desire, as yet undeveloped, to establish *once and for all* a solid theoretical base on which to construct the "art of building".

Historiography in the air

Wotton's few references to Baldi's work, but above all the "novelty" of the approach to the theory of vaults had immediate repercussions in England.[14] John Evelyn's judgement, which appears in the 1664 English edition (Fréart 1664) of the famous *Parallèle de l'architecture antique avec la moderne* of 1650 by Roland Fréart de Chambray (Fréart 1650), underlines the great impression made by the theorems, so much so that he speaks of the work as a whole in these terms: "Sir H. Wotton in his concise and useful *Theorems*" (Fréart 1664, 118). Further, it is very likely that the *Elements* influenced Christopher Wren, both directly and through his father, even if the way the problem of the thrusts of the vaults is set forth as found in the notes published in *Parentalia* (Wren 1750) does not reveal a significant connection to Baldi's doctrine.

Because of this predictable continuity of interest, it is English historiography which has most occupied itself with the *Exercitationes*, both for obvious reasons and in direct relation to Wotton. Up to the present day, however, scholars of construction history have only attended to it superficially; no research has been dedicated to an exhaustive parallel reading of Baldi and Wotton, even though the strict dependence of the *Elements* on the *Exercitationes* is obvious even to a distracted reader.

This appears curious in light of the significance of Baldi's text in the context of the times, because of its contents regarding mechanics as well as the originality of its aperture to the architectural front; indeed, it must have been the *mixed* nature of the treatment itself that has given rise to its negligence on the part of scholars. The text is too architectural for scholars of mechanics, and too mechanical for the architects, with the result that neither of the two groups has deemed it worthy of attention, or, if they have done so, this attention has been limited to a superficial reading. To this is added the basic linguistic difficulty of Baldi's text: his Latin text requires a thorough knowledge of that language. This detail appears to have led the historians to limit themselves to a reference, or to ask for occasional help from translators,[15] without tackling the work as a whole. The few direct references to the text do not in fact take into account the treatise as a whole, most likely because of the difficulties faced by the very translators and linguists called in to help, who grapple with a prose that is not always easy to read and with an edition plagued by numerous printing errors (in particular as regards references to the figures within the text).

As regards contemporary histories dedicated to Wotton's theorems, reference is usually made to the unpublished thesis of 1970 by Harold Dorn (Dorn 1970), entitled *The Art of Building and the Science of Mechanics*, but the essay that truly opened the way is the 1968 thesis by Marilyn P. Caldwell (Caldwell 1968), also

unpublished, entitled *Sir Henry Wotton: Aspects of English Taste in the Early Seventeenth Century*, which presents a careful investigation of Wotton and the sources for the *Elements*. Two years later, Dorn (who in all probability did not know of Caldwell's work) takes up the argument, providing the first "technical" reading of Wotton's work, and, in perspective, of some of Baldi's. However, in this case as well, Baldi's *Exercitationes* are not carefully read, but hastily thumbed through.

The works of Caldwell and Dorn remain apparently little known to scholars of both mechanics and mechanics applied to architecture.[16] In 1986 did an article by David Yeomans look at the relationships between Wotton and Baldi in detail[17] (Yeomans 1986), but only with regards to the subject of trusses (in the same way that Dorn limited himself to vaults), and the corresponding, fleeting, citation by Wotton of Baldi (Wotton 1624, 79). The subject as a whole remains unexplored. Further, Yeomans seems to have read only a few lines of Baldi, so few that sometimes he misses the point of his arguments (Valeriani 2005).

Thus we have a series of lost opportunities. It leads one to think that none of the authors cited here have attentively studied the fundamental historic essays of Heyman, beginning in the 1960s. If they had done so, their analyses would have been more rigorous and reliable, and Wotton's *Exercises* on Baldi's *Exercitationes* would have found better prepared interpreters. Heyman had suggested methods and ways, and it is surprising even today how few seem to have understood them. This is demonstrated by the article published in 2000 by Sergio Luis Sanabria entitled *Perils of Certitude in the Structural Analysis of Historic Masonry Buildings* (Sanabria 2000) in which he reviews Heyman's book of 1996, *Arches, Vaults and Buttresses* (Heyman 1996) and presents an analysis of the historiography of structural mechanics that is approximate and full of gaps. The paper ends peremptorily: "The intellectual horizon of this book is fixed in the 1960s".

In actual fact, Heyman's work cleared the way for the 1960s and 1970s and permitted historians to look ahead. If even today there are those who have not understood this, it is because his writings were and are "out of date": projected into the future, forgetful of the *gravitas* of the present. Sanabria, with his eyes fixed on the manuscript by Rodrigo Gil de Hontañón, was not able to comprehend this. Another lost opportunity, the fruit of an immature historiography.

Notes

1. The year given, here as elsewhere, refers to the Gregorian calendar.
2. From the dedicatory letter to the Prince of Wales, dated 1624. See (Smith 1907, II: 284–285).
3. Wotton was "Italianate" to the point of correctly speaking Italian and being jokingly referred to as "Signor Fabritio". See (Smith 1907, 1: 126).

4. It is not possible within the scope of the present paper to present a detailed comparison between the texts of Baldi and Wotton. This is the subject of my current research, to be published shortly, which will present an in-depth reading of Baldi's and Wotton's works first presented in (Becchi 2004).
5. Baldi's work is inserted in the long tradition of Renaissance comments on *Mechanical Problems* (at that time attributed to Aristotle). For more on this, see (Becchi 2004).
6. "Dubitatur, quare, quo longiora sunt ligna, tanto imbecilliora fiant, & si tolluntur, inflectuntur magis: tametsi quod breue est ceu bicubitum fuerit, tenue, quod vero cubitorum centum crassum?"
7. "Modo vt ex hac contemplatione, quæ alias inutilis videtur, aliquam vtilitatem capiamus, & ex his quæ contemplabimur, Architecti prudentiores fiant, ist hæc ipsa, de quibus agimus, ad rem ædificatoriam commode aptabimus".
8. This observation is only partially true in Baldi's case, who had occupied himself at length with "built architecture" in contexts which are little known even today (Becchi 2004, 74).
9. The letter, dated December 1621, is kept at the Biblioteca Oliveriana (Pesaro), ms. 413, cc. 11–12. It is republished in with some differences in transcription in Gamba and Montebelli (1988, 187–189).
10. "Se l'è passata assai scuram.te da certi luoghi in poi poco degni dell'ingegno suo che è saltato da una cosa all'altra et afferrate certe occasioni si fatte come per essempio alla questi XVI che s'imagina nella rottura delle volte con moto circolare, et non è vero, et ha fatto per una bagatella un rumor d'importanza".
11. The asterisk indicates notes printed in the margin, which are given here in square brackets.
12. In *De architectura*, book 4, chap. 8, Vitruvius writes: "Cum in his rebus animadversum fuerit, uti ea diligentia in his adhibeatur, non minus etiam observandum est, uti omnes structurae perpendiculo respondeant neque habeant in ulla parte proclinationes".
13. Casati mistakenly cites Vitruvius, *De Architectura*, bk. 4, chap. 2, but literally quotes chap. 8.
14. That equal attention was paid in Germany is testified to by the volume by Daniel Mögling (Mögling 1629), which contains a faithful translation of Baldi's comment on *Quaestio* XVI.
15. Dorn openly and honestly declared his dependence on translators: "I wish to thank Mr Eugene Sabini for assisting me with the translation of this section of Baldi's book" (Dorn 1970, 53, note 1).
16. The Baldi's theory of arches is not analysed in the principal papers on the history of structural mechanics published in the 1980s and 1990s, although the thesis of Caldwell had already been cited in the fundamental work of Johannes Dobai (1974, 1: 369 and 458).
17. See also (Yeomans 1984) and (Yeomans 1992).

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Analysis of continuous prestressed concrete beams

Christopher J. Burgoyne

This conference is devoted to the development of structural analysis rather than the strength of materials, but the effective use of prestressed concrete relies on an appropriate combination of structural analysis techniques with knowledge of the material behaviour. Design of prestressed concrete structures is usually left to specialists; the unwary will either make mistakes or spend inordinate time trying to extract a solution from the various equations.

There are a number of fundamental differences between the behaviour of prestressed concrete and that of other materials. Structures are not unstressed when unloaded; the design space of feasible solutions is totally bounded; in hyperstatic structures, various states of self-stress can be induced by altering the cable profile, and all of these factors get influenced by creep and thermal effects. How were these problems recognised and how have they been tackled?

Ever since the development of reinforced concrete by Hennebique at the end of the 19th century (Cusack 1984), it was recognised that steel and concrete could be more effectively combined if the steel was pretensioned, putting the concrete into compression. Cracking could be reduced, if not prevented altogether, which would increase stiffness and improve durability. Early attempts all failed because the initial prestress soon vanished, leaving the structure to behave as though it was reinforced; good descriptions of these attempts are given by Leonhardt (1964) and Abeles (1964).

It was Freyssinet's observations (Freyssinet 1956) of the sagging of the shallow arches on three bridges that he had just completed in 1913 over the River Allier near Vichy which led eventually to his patent on prestressed concrete (Freyssinet

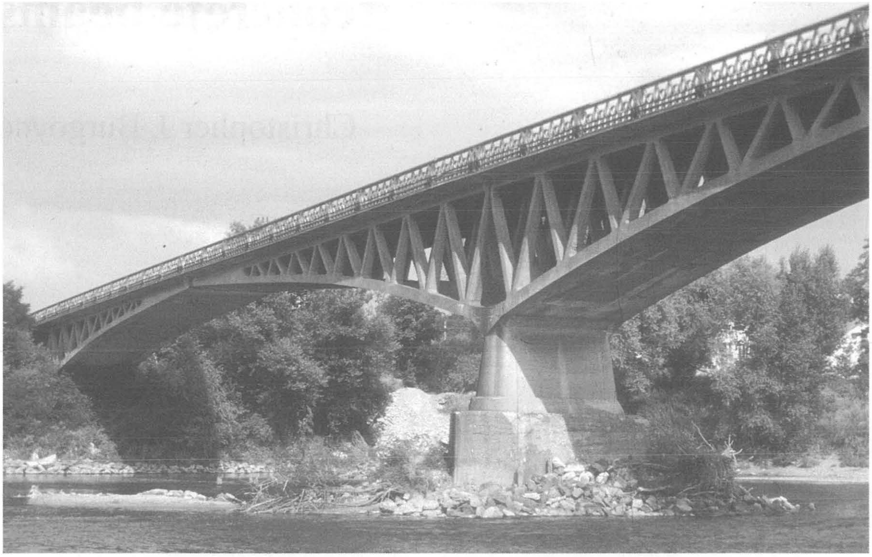

Figure 1
Boutiron Bridge, Vichy

1928). Only the bridge at Boutiron survived WWII (Fig. 1). Hitherto, it had been assumed that concrete had a Young's modulus which remained fixed, but he recognised that the deferred strains due to creep explained why the prestress had been lost in the early trials. Freyssinet (Fig. 2) also correctly reasoned that high tensile steel had to be used, so that some prestress would remain after the creep had occurred, and also that high quality concrete should be used, since this minimised the total amount of creep. The history of Freyssinet's early prestressed concrete work is written elsewhere (Grote and Marrey 2000).

At about the same time work was underway on creep at the BRE laboratory in England (Glanville (1930 and 1933). It is debatable which man should be given credit for the discovery of creep but Freyssinet clearly gets the credit for successfully using the knowledge to prestress concrete.

There are still problems associated with understanding how prestressed concrete works, partly because there is more than one way of thinking about it. These different philosophies are to some extent contradictory, and certainly confusing to the young engineer. It is also reflected, to a certain extent, in the various codes of practice.

Permissible stress design philosophy sees prestressed concrete as a way of avoiding cracking by eliminating tensile stresses; the objective is for sufficient

Figure 2
Eugene Freyssinet

compression to remain after creep losses. Untensioned reinforcement, which attracts prestress due to creep, is anathema. This philosophy derives directly from Freyssinet's logic and is primarily a working stress concept.

Ultimate strength philosophy sees prestressing as a way of utilising high tensile steel as reinforcement. High strength steels have high elastic strain capacity, which could not be utilised when used as reinforcement; if the steel is pretensioned, much of that strain capacity is taken out before bonding the steel to the concrete. Structures designed this way are normally designed to be in compression everywhere under permanent loads, but allowed to crack under high live load. The idea derives directly from the work of Dischinger (1936) and his work on the bridge at Aue in 1939 (Schönberg and Fichter 1939), as well as that of Finsterwalder (1939). It is primarily an ultimate load concept. The idea of partial prestressing derives from these ideas since the addition of quite significant amounts of untensioned reinforcement does not alter the logic (Emperger 1939).

The *Load-Balancing philosophy*, introduced by T. Y. Lin, uses prestressing to counter the effect of the permanent loads (Lin 1963). The sag of the cables causes

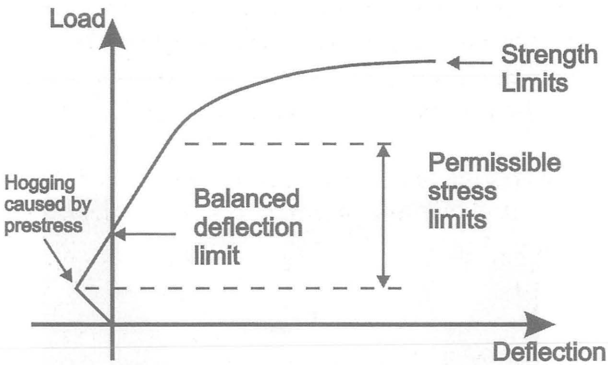

Figure 3
Load deflection curve

an upward force on the beam, which counteracts the load on the beam. Clearly, only one load can be balanced, but if this is taken as the total dead weight, then under that load the beam will perceive only the net axial prestress and will have no tendency to creep up or down.

These three philosophies all have their champions, and heated debates take place between them as to which is the most fundamental.

Section design

From the outset it was recognised that prestressed concrete has to be checked at both the working load and the ultimate load. For steel structures, and those made from reinforced concrete, there is a fairly direct relationship between the load capacity under an *allowable stress* design, and that at the ultimate load under an *ultimate strength* design. Older codes were based on permissible stresses at the working load; new codes use moment capacities at the ultimate load. Different load factors are used in the two codes, but a structure which passes one code is likely to be acceptable under the other.

For prestressed concrete, those ideas do not hold, since the structure is highly stressed, even when unloaded. A small increase of load can cause some stress limits to be breached, while a large increase in load might be needed to cross other limits. The designer has considerable freedom to vary both the working load and ultimate load capacities independently; both need to be checked.

A designer normally has to check the tensile and compressive stresses, in both the top and bottom fibre of the section, for every load case. The critical sections

are normally, but not always, the mid-span and the sections over piers but other sections may become critical when the cable profile has to be determined.

The stresses at any position are made up of three components, one of which normally has a different sign from the other two; consistency of sign convention is essential.

If P is the prestressing force and e its eccentricity, A and Z are the area of the cross-section and its elastic section modulus (top or bottom fibres), while M is the applied moment, then

$$f_t \leq \frac{P}{A} + \frac{Pe}{Z} - \frac{M}{Z} \leq f_c \tag{1}$$

where f_t and f_c are the permissible stresses in tension and compression.

Thus, for any combination of P and M, the designer already has four inequalities to deal with.

The prestressing force differs over time, due to creep losses, and a designer is usually faced with at least three combinations of prestressing force and moment;

- he applied moment at the time the prestress is first applied, before creep losses occur,
- the maximum applied moment after creep losses, and
- the minimum applied moment after creep losses.

Other combinations may be needed in more complex cases. There are at least twelve inequalities that have to be satisfied at any cross-section, but since an I-section can be defined by six variables, and two are needed to define the prestress, the problem is over-specified and it is not immediately obvious which conditions are superfluous. In the hands of inexperienced engineers, the design process can be very long-winded. However, it is possible to separate out the design of the cross-section from the design of the prestress. By considering pairs of stress limits on the same fibre, but for different load cases, the effects of the prestress can be eliminated, leaving expressions of the form:

$$Z \geq \frac{\textit{Moment Range}}{\textit{Permisible Stress Range}} \tag{2}$$

These inequalities, which can be evaluated exhaustively with little difficulty, allow the minimum size of the cross-section to be determined.

Once a suitable cross-section has been found, the prestress can be designed using a construction due to Magnel (figure 4). The stress limits can all be rearranged into the form:

$$e \geq -\frac{Z}{A} + \frac{1}{P}(fZ + M) \tag{3}$$

Figure 4
Gustave Magnel

By plotting these on a diagram of eccentricity versus the reciprocal of the prestressing force, a series of bound lines will be formed. Provided the inequalities (2) are satisfied, these bound lines will always leave a zone showing all feasible combinations of P and e. The most economical design, using the minimum prestress, usually lies on the right hand side of the diagram, where the design is limited by the permissible tensile stresses.

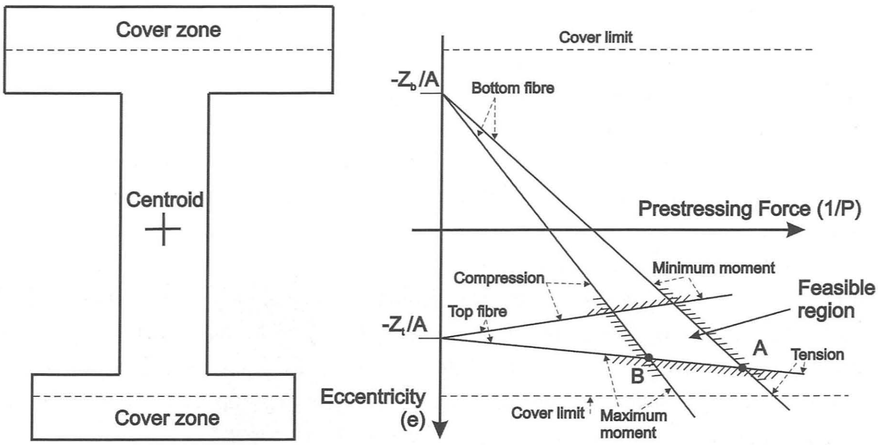

Figure 5
Magnel diagram

Plotting the eccentricity on the vertical axis allows direct comparison with the cross-section, as shown in figure 5. Inequalities (3) make no reference to the physical dimensions of the structure, but these practical cover limits can be shown as well.

A good designer knows how changes to the design and the loadings alter the Magnel diagram. Changing both the maximum and minimum bending moments, but keeping the range the same, raises and lowers the feasible region. If the moments become more sagging the feasible region gets lower in the beam. In general, as spans increase, the dead load moments increase in proportion to the live load. A stage will be reached where the economic point (A on figure 5) moves outside the physical limits of the beam; Guyon (1951a) denoted the limiting condition as the *critical span*. Shorter spans will be governed by tensile stresses in the two extreme fibres, while longer spans will be governed by the limiting eccentricity and tensile stresses in the bottom fibre (assuming sagging bending). However, it does not take a large increase in moment for point B to move outside the cover limit, at which point compressive stresses will govern in the bottom fibre under maximum moment.

Only when much longer spans are required, and the feasible region moves as far down as possible, does the structure become governed by compressive stresses in both fibres.

Continuous beams

The design of statically determinate beams is relatively straightforward; the engineer can work on the basis of the design of individual cross-sections, as outlined above. A number of complications arise when the structure is indeterminate which means that the designer has to consider, not only a critical section, but also the behaviour of the beam as a whole. These are due to the interaction of a number of factors, such as Parasitic Moments, Creep, Temperature effects and Construction Sequence effects. It is the development of these ideas which forms the core of this paper. The problems of continuity were addressed at a conference in London (Andrew and Witt 1951). The basic principles, and nomenclature, were already in use, but to modern eyes concentration on hand analysis techniques was unusual, and one of the principle concerns seems to have been the difficulty of estimating losses of prestressing force.

Secondary Moments

A prestressing cable in a beam causes the structure to deflect. Unlike the statically determinate beam, where this motion is unrestrained, the movement causes a redistribution of the support reactions which in turn induces additional moments. These are often termed Secondary Moments, but they are not always small, or Parasitic Moments, but they are not always bad. They are normally denoted by M_2.

Freyssinet's bridge across the Marne at Luzancy, started in 1941 but not completed until 1946, is often thought of as a simply supported beam, but it was actually built as a two-hinged arch (Harris 1986), with support reactions adjusted by means of flat jacks and wedges which were later grouted-in (Fig. 6). The same principles were applied in the later and larger beams built over the same river.

Magnel built the first indeterminate beam bridge at Sclayn, in Belgium (Fig. 7) in 1946. The cables are virtually straight, but he adjusted the deck profile so that the cables were close to the soffit near mid-span but above the centroidal axis at the internal support (Magnel 1951). Even with straight cables the sagging secondary moments are large; about 50% of the hogging moment at the central support caused by dead and live load.

The nomenclature for dealing with these reactant moments was quickly established (Guyon 1951b). The designer needed to distinguish between the actual location of the *cable profile* (e_s), and its apparent position, known as its *line of thrust* (e_p). The two profiles differ by M_2/P. The cable profile has to fit within the section and is subject to cover constraints, but it is the line of thrust which has to be used in the stress calculations.

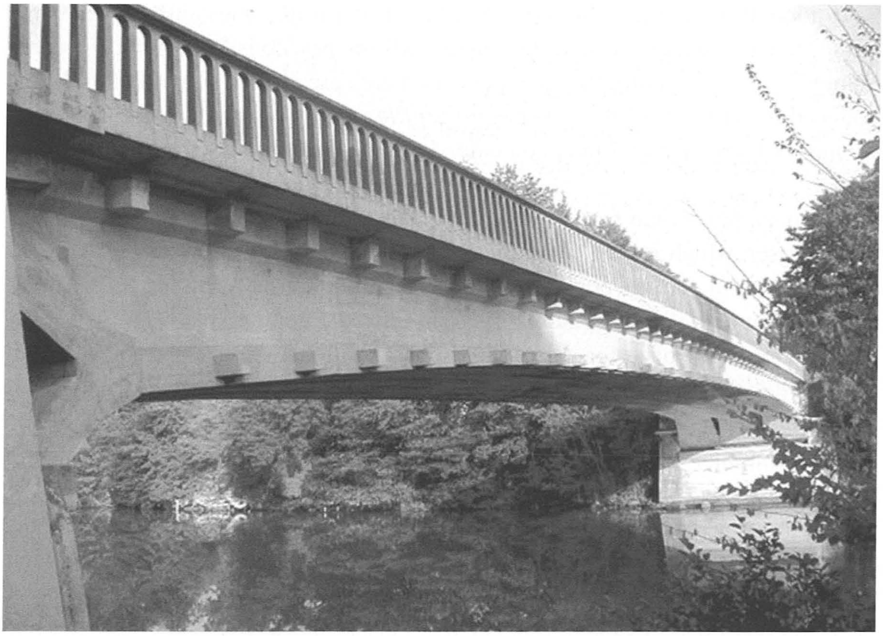

Figure 6
Luzancy Bridge

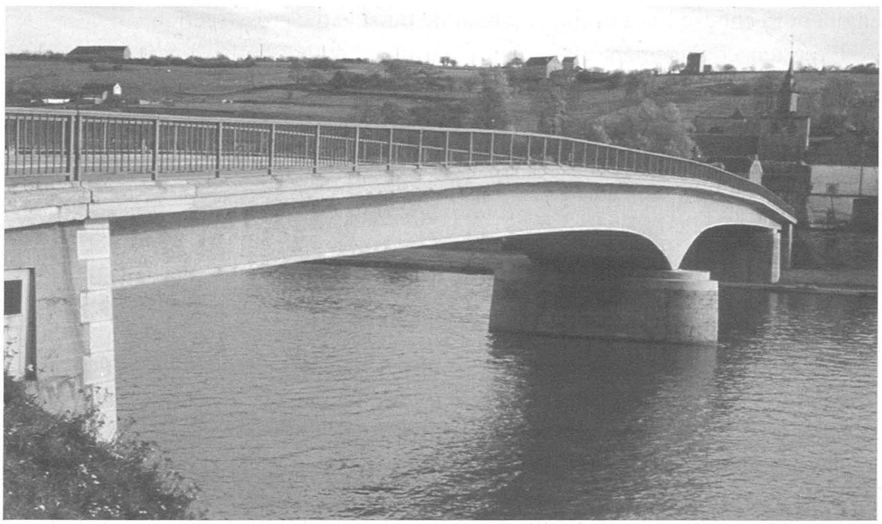

Figure 7
Sclayn Bridge

A good designer can exploit this freedom, but it is also the cause of problems; the secondary moments cannot be found until the profile is known but the cable cannot be designed until the secondary moments are known. Guyon (1951b) introduced the concept of the *concordant profile*, which is a profile that causes no secondary moments; and thus coincide. Any line of thrust is itself a concordant profile.

The designer is then faced with a slightly simpler problem; a cable profile has to be chosen which not only satisfies the eccentricity limits (3) but is also concordant. That in itself is not a trivial operation, but is helped by the fact that the bending moment diagram that results from *any* load applied to a beam will itself be a concordant profile for a cable of constant force. Such loads are termed *notional loads* to distinguish them from the real loads on the structure. Superposition can be used to progressively build-up a set of notional loads whose bending moment diagram gives the desired concordant profile. The question of whether such a profile *can* be found will be addressed later.

Inexperienced designers often stop at this stage of the design process, but they are clearly then not making full advantage of the power of prestressing, and their structures are often uneconomic. Guyon pointed out that it was possible to move the cable profile by means of a *linear transformation* in such a way that the line of thrust remains unchanged. Having chosen a suitable concordant profile (which remains as the line of thrust), the designer can then alter the profile by linear transformations until a suitable cable profile is achieved. This freedom allows the engineer to choose lines of thrust which lie outside the cross-section.

Experienced designers often claim that they "do not bother" with concordant profiles; they simply use their judgement to choose a secondary moment that they expect to obtain in the particular structure they are designing. They can then use modified forms of the eccentricity equations (3) which allow them to produce limits on the actual cable profile directly:

$$e_s \geq -\frac{Z}{A} + \frac{1}{P}(fZ + M + M_2) \tag{4}$$

A slightly different problem now has to be solved; how to find a cable profile that not only satisfies the local eccentricity limits but also generates the required value of.

The two methods are, in fact, directly equivalent, since a profile that satisfies one set of constraints will automatically satisfy the other.

Analysis to determine M_2

The calculation of the secondary moments M_2 or the determination of the line of thrust e_p, which are equivalent, can be done in several ways. The development of these methods reflects changes elsewhere in analysis techniques, and of course the adoption of computer techniques. At the 1951 conference, for example, Guyon proposed the method of nodal points (a variation of the method of fixed points) (Guyon 1951b); as with most methods at the time the objective was to minimise the number of simultaneous equations to be solved. The most common method in use today is to determine the forces that the cable exerts on the concrete and then to analyse the beam under those loads. This can be done at several levels of detail; if the structure is analysed as a beam the resulting bending moment will be the total effect of the cable (= Pe_p) and the reactions will be those that contribute to the secondary moments M_2. If a more detailed method, such as a finite element analysis, is used, the distribution of cables across the width can be determined, as can local effects of the cable profile distribution, which can be particularly important with cables which have significant horizontal curvature.

Alternatively, for continuous beams, use can be made of virtual work to derive a set of equations from which the secondary moments can be determined directly in terms of the actual cable profile (see, for example, Burgoyne 1988). This is not usually much easier than the cable force method, but it does allow analytical formulations to be developed which can be used to derive further theories.

Existence of line of thrust?

Experienced engineers adopt their own strategies for designing complex structures. Low Low, for example, showed that there were limits on the minimum prestressing force that is needed in a continuous beam. One minimum limit was derived directly from the Magnel diagram, while another considered the range of eccentricities that have to be allowed for between the maximum sagging region at mid-span and the maximum hogging region over the piers. If the prestressing force is not high enough, the eccentricity range is too large and it is impossible to find a secondary moment that leaves the cable profile always inside the structure.

But Low also showed that there was a third limit on the prestressing force which had to be satisfied before a solution could be found, which he called the "third equation". It was later shown (Burgoyne 1988) that this limit related to the *existence* of the line of thrust. If the prestressing force is too low, then a cable placed anywhere between the upper or lower limits on the cable profile will give secondary moments of the same sign. Under these conditions, no concordant pro-

file can exist, so the designer will never be able to find a satisfactory solution without increasing the prestressing force. By satisfying Low's third limit, the designer is assured that a valid profile exists, even if it still has to be found.

Applicability of plastic theory

Prestressed concrete beams are normally checked for ultimate moment capacity, but that is not the same thing as saying that plastic theory can be used to design such beams. Plastic theory can only be used if prestressed concrete structures have sufficient ductility to allow redistribution of bending moments as hinges form.

La Grange conducted a study on indeterminate beams and frames, where he concluded that indeterminate prestressed concrete structures attained a load at failure which was just below that predicted by full plastic theory (La Grange 1961). The small discrepancy was due to the post-peak softening that occurs in prestressed concrete structures, so that in order to allow the full set of plastic hinges to develop, the first hinges undergo some reduction in moment capacity. However he concluded that the difference was, in practical terms, negligible.

Prestressing steel is much stronger than normal reinforcing steel and does not exhibit a well-defined yield point, so the steel has to have much higher strains before significant plastic deformation occurs. Some of that strain capacity is consumed during the act of prestressing but significant elastic curvatures still have to take place before yielding can occur. This gives a lower limit on the depth of the neutral axis at failure.

There is a further complication, because higher strength steels are typically more brittle than reinforcing steels, with strain capacities of the order of 3 %. Thus designers should limit the strains that develop in the steel at the ultimate load, and a limit which is frequently applied is to limit the *additional* strain, after prestressing, to 1 %. This limit can be criticised as too low, but it takes account of the fact that the analysis at the ultimate load uses an average strain along the tendon, while the strain at crack locations can be higher. This condition provides an upper limit on the depth of the neutral axis.

The result of these two conditions is that prestressed concrete beams can only behave plasticly if they satisfy relatively narrow limits on the position of the neutral axis, which in turn provides narrow limits on the section geometry. Codes of practice normally aim to force designs into this narrow band of acceptability, and they only allow redistribution if certain conditions on the neutral axis position are satisfied.

Secondary moments at the ultimate load?

A closely related question for the designer of continuous prestressed concrete is whether secondary moments should be taken into account when the ultimate moment is calculated. The question is not trivial; secondary moments can be of the same order of magnitude as the dead-load bending moments, although distributed in a different way.

The logic behind limit-state codes is to check each possible failure mechanism of the structure, such as cracking, vibration or collapse. A proper check on the ultimate limit-state would therefore require determination of the final collapse mechanism of the structure. When the final plastic hinge forms the structure becomes a mechanism, so when the penultimate hinge formed, the structure must have been statically determinate; secondary moments do not exist in statically determinate beams. The moment distribution in the beam can be found purely by equilibrium considerations which will differ from the elastic moments by a certain amount of redistribution.

However, in most cases, ultimate moment capacities are checked on a section-by-section basis by applying factored values of the elastic load distribution. Some codes make no mention of secondary moments, but others allow the inclusion of M_2 in the ultimate load calculation (Mattock 1983). In effect, this condition ensures that the first plastic hinge forms with a sufficient reserve of strength; up to this load, the structure has been behaving elastically, so secondary moments would, indeed, have been present. In a beam with sagging secondary moments the effect can be to significantly reduce the ultimate moment capacity that has to be provided over the piers, and to increase the moment capacity that is required in the span regions.

There have been some laboratory studies of continuous beams where the support reactions were measured as loads were increased until a collapse mechanism developed. The magnitudes of these reactions allowed the presence (or absence) of the secondary moments to be monitored, and the results showed that the secondary moments disappeared as the final hinge formed. The plastic hinges did not form suddenly, but slowly developed through an elasto-plastic regime (Mattock et al. 1971).

The existence of secondary moments is an academic question if the structure is ductile, since the moment distribution, with or without M_2, satisfies the Lower Bound Theorem if the structure is provided with adequate rotation capacity, but codes do not always allow these effects to be taken into account, or limit the amount of redistribution that can take place.

Temperature effects

Temperature variations apply to all structures but the effect on prestressed concrete beams can be more pronounced than in other structures. The temperature profile through the depth of a beam (Emerson 1973) can be split into three components for the purposes of calculation (Hambly 1991). The first causes a longitudinal expansion, which is normally released by the articulation of the structure; the second causes curvature which leads to deflection in all beams and reactant moments in continuous beams, while the third causes a set of self-equilibrating set of stresses across the cross-section.

The reactant moments can be calculated and allowed-for, but it is the self-equilibrating stresses that cause the main problems for prestressed concrete beams. These beams normally have high thermal mass which means that daily temperature variations do not penetrate to the core of the structure. The result is a very non-uniform temperature distribution across the depth which in turn leads to significant self-equilibrating stresses. If the core of the structure is warm, while the surface is cool, such as at night, then quite large tensile stresses can be developed on the top and bottom surfaces. However, they only penetrate a very short distance into the concrete and the potential crack width is very small. It can be very expensive to overcome the tensile stress by changing the section or the prestress, and they are normally taken into account by the provision of a mesh of fine bars close to the surface.

A larger problem can arise if thermal stresses act as a trigger for more damaging cracking, such as the release of locked-in heat of hydration effects which can occur when a thick web is associated with thin slabs.

Construction sequence effects

Prestressed concrete tends to be used for the longer-span bridge structures, which often means that they are built sequentially. As a result, the bending moments at the end of construction differ from those which would be expected if the bridge had been built in one go (the *monolithic moment*). As an example, balanced cantilever construction builds out from a central pier, so the structure is inevitably in hogging bending throughout. When the tips of two cantilevers meet they are joined by an in-situ stitch, or sometimes by a short suspended span that is usually made fully continuous. The cantilever will have prestressing cables at the top to resist hogging bending, while continuity cables will be introduced across the joint to resist the sagging bending that will occur later.

The designer has to allow for the temporary condition, and also for the *trapped moments* that are induced by the construction sequence. These trapped

moments can be large, and obey the same rules as the secondary moments, in that they are brought about by a redistribution of the dead-load support reactions. The designer may deliberately choose to use the continuity cables to induce a secondary moment that reduces the trapped moment.

Further trapped moments can be induced by the use of temporary prestressing cables which are introduced when the structure is in one configuration, and then removed later after the support conditions have changed. For example, in span-by-span construction, where a long viaduct is built one span at a time, it is sometimes necessary to introduce temporary cables to resist sagging bending moments that occur during construction but which will be removed later. Putting a cable into a two-span structure (for example), and then removing it once the structure is more indeterminate, does not leave a zero stress state; these effects should not be overlooked.

Creep effects

The final effect that needs to be considered is, appropriately enough, creep (Bazant and Wittmann 1982). It was Freyssinet's original observation of creep that made prestressed concrete possible since he managed to reduce the loss of force caused by creep. In simply supported beams creep causes some loss of prestress and increased deflections, which may need to be taken into account, but it does not alter the distribution of bending moments so the design remains relatively straightforward.

If the structure is indeterminate there is always the possibility that the bending moments may be altered by redistribution of the support reactions. If the structure is built in one piece, all the concrete will be of the same age, and its effective modulus will change uniformly throughout the structure. No redistribution of forces is to be expected under these circumstances.

However, if the concrete is of different ages, the amount of creep that can occur in the various parts of the structure will vary, which allows redistribution of moments. It is now well-established that the structure will creep towards the monolithic state, and the designer can take the as-built condition (including trapped moments) and the monolithic state as limiting conditions for the behaviour of the beam. This simplifies the design process.

England has studied the effect of temperature variation through the depth of the beam. Creep is temperature dependent and takes place more quickly on the warmer side of a structure than on the colder side, which can significantly alter the load distribution. This work was originally applied to nuclear reactor containment vessels, where the temperature variation across the thickness can be of the order of 100° C (England et al. 1984). The work makes use of the concept of a

steady-state, when creep can continue but without redistribution of stress. More recently, it has been shown that the much smaller temperature variations that can be expected through the depth of a bridge deck, which may be of the order of 5° C, can also have a significant effect. The speed with which creep occurs is very heavily dependent on the relative ages of the concrete in different parts of the structure (Xu and Burgoyne 2005).

Conclusion

The successful design of continuous prestressed concrete beams cannot be divorced from the techniques used to analyse the structure, and the way these have developed in the 60 years since the first indeterminate structures were built is a fascinating reflection on the way structural analysis has developed over the same period.

It remains the case that designers cannot blindly use analysis programs without fundamental understanding of the way prestressed concrete behaves.

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A preliminary structural analysis of a Guastavino spiral staircase shell

Christopher R. Calladine

The Rafael Guastavinos, father and son (1842–1908; 1872–1950) developed successfully in the United States the traditional catalan method for constructing thin shells from strong ceramic tiles and rapidly-hardening gypsum plaster and "hydraulic cement" (Collins 1968, Huerta 2003). Among their many domes the largest, with a span of 33 m, covered the crossing of the Cathedral of St John the Divine, New York. They constructed such domes without the use of scaffolding. Instead the masons, working at the free edge of the previous day's construction, would lean over and move the edge forwards 18 inches by adding new tiles; and this fresh construction would then bear their weight on the following day. Typically the terracotta tiles measured 9 by 18 inches, and were 1 inch thick (230 × 460 × 25 mm).

The Guastavinos called such shells "timbrel vaults", because they resounded like a drum or tambourine when jumped upon or slapped. The Guastavinos were at pains to distinguish their vaults from the classical, mediaeval style of vault, in which the component masonry blocks were full-thickness "voussoirs", with dry or weak-mortar interfaces. Although it may have been possible to advance the construction of mediaeval vaults between ribs without the use of much formwork, such shells were less "cohesive" —another term used frequently by the Guastavinos— than their "timbrel" vaulting. Provided the Guastavino cement is strong enough, the multiple-layer overlapping-tile construction of their vaults is more akin to a heavy-duty *papier maché* than to a single-layer mosaic of voussoirs.

The "Guastavino system" was also used to build staircases of several different kinds (Collins 1968, Huerta 2003). In this paper I shall attempt a structural

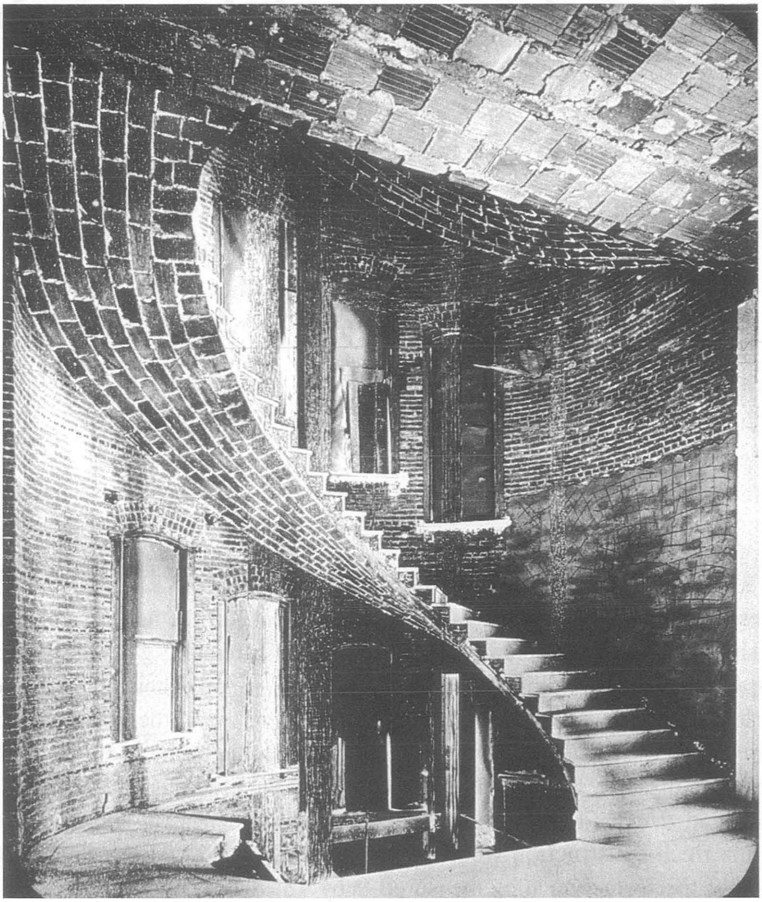


Figure 1
Guastavino spiral staircase in the First National Bank of Paterson, New Jersey, ca. 1890. The soffit layer of tiles is arranged circumferentially. This staircase is elliptical in plan, unlike those analysed in this paper, which have circular planforms. From Collins, 1968, figure 23A

analysis of their simple *spiral* staircases, which were built inwards from a groove in a cylindrical outer wall, and had a free inner edge. Figure 1 shows an example of such a staircase under construction; and some staircases of this kind still survive.

After the outer cylindrical wall had been constructed, the next stage was to build a spiral, or helicoidal, shell; then, finally, the steps themselves could be

added —apparently as non-structural units, made from tiles or masonry. The workers would build up the spiral shell from below, standing or sitting on temporary supports (Ochsendorf 2005). Geometric accuracy was ensured by a system of specially-placed taut strings. Tiles in the lowest —or "soffit"— layer were sometimes laid radially, in straight lines of tapering tiles; but sometimes —as in figure 1— they were laid in parallel, circumferential rows (Collins 1968, figure 23A, C). In any case it was not necessary to cut the tiles to shape with great precision, whether for a dome or a spiral staircase, since the plaster of Paris filled the gaps in the lowest layer (Fig. 1); and the next layer of tiles would be staggered in order to cover the joints, and its gaps would be filled by the cement.

For the purposes of structural analysis I shall therefore model the staircase as a uniform thin helicoidal shell made from elastic material. My aim will not be to analyse the performance of any particular staircase: rather, I shall try to discover the basic mode of structural action of these shells, and how it depends on the leading geometrical parameters of the staircase. In order to make the problem more tractable, I shall confine attention to the thin helicoidal timbrel shell itself, instead of the finally completed staircase.

Now there are many papers in the literature on the analysis of elastic helicoidal shells: see, for example, Hu et al. (2005) and references therein. But almost all of them are concerned with twisted turbine blades, which obviously have a very different shape from spiral staircases. Also, such papers usually contain heavy mathematical manipulations, of a kind which seems out-of-place for an initial structural assessment of a staircase-shell.

Preliminary matters

The geometry of a sector of a generic spiral staircase is shown in figure 2. The outer rigid wall is a right cylinder of radius b, while the inner edge lies on a concentric cylinder of radius a. The obvious co-ordinate system is r, θ, z, as shown; and the spiral surface has the equation

$$z = c\theta, \tag{1}$$

where c is a (negative) constant.

Vertical planes through the axis (i. e. planes θ = constant) intersect the surface in horizontal lines.

For present purposes it is convenient to define the "climb-angles" of the inner and outer edges as α, β respectively, as marked in figure 2; and it follows that the four defining parameters a, α, b, β are related by the condition

$$a \tan \alpha = b \tan \beta. \tag{2}$$

Now the classical theory of shells can take many different forms, depending on the precise way in which bending and stretching effects interact in the performance of the shell as a structure under load. It is reasonable to assume that bending plays a relatively unimportant part of the action of timbrel spiral staircases —I give an order-of-magnitude argument below— and so we shall be concerned here almost entirely with membrane effects.

There are, basically, two approaches to the "membrane" analysis of shells. One approach is to set up the "membrane equations of statical equilibrium" (e. g. Timoshenko and Woinowsky-Krieger 1959, chap. 14; Calladine 1983, chap. 4, 7) and to solve them subject to appropriate load and boundary conditions. Such

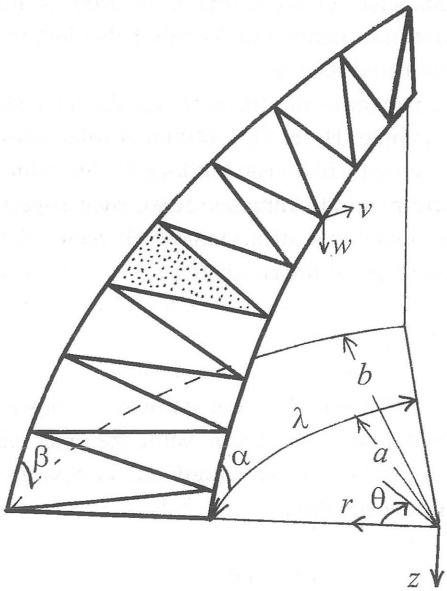


Figure 2
General, perspective view of a helicoidal staircase-shell, idealised here as a chain of narrow triangular facets. The lower part of the rigid cylindrical supporting wall, of radius b, is shown; and the inner free edge lies on a cylinder of radius a. "Climb" angles β and α are shown, as is the angle λ subtended by the helicoidal sector in plan, and the (r, θ, z) co-ordinate system. There are two kinds of triangular facet: outer (one of them is shaded) and inner. At the top and bottom of the sector there are half-triangles. Components of displacement w, v (in the z, θ directions) are also shown at one inner node

equations may have a unique solution —that is, the structure may be statically determinate— but that will depend on boundary conditions and other matters. Then the strains may be deduced by application of Hooke's law; and the displacements found by solution of the kinematic strain-displacement equations. The other main approach is to assume a mode or pattern of small displacements of the surface; then to deduce a corresponding pattern of strains by means of the equations of kinematical compatibility; and finally to perform an energy-balance between the strain energy of distortion of the structure and the loss of potential energy of the applied loads. For the present problem the second approach seems to be more direct, and we shall pursue it. But we shall also make some remarks later on touching the statical equilibrium of the shell.

Analysis, Stage 1

Figure 2 is actually a drawing of part of a physical model which I made out of cardboard. A smooth helicoidal surface, as described above, has negative Gaussian curvature (e. g. Calladine 1983, chap. 5), and so it cannot be made by development from a flat sheet of cardboard. However, a fair approximation to the helicoidal surface can be made from a sheet of cardboard as a chain of narrow, plane triangles, as illustrated. The small-angle creases or folds between successive "outer" and "inner" plane facets are alternately positive and negative.

It is immediately obvious from a physical model of this kind that the surface is not necessarily *rigid*; for if the upper and lower ends of the sector of the helicoidal surface are not fixed, the entire inner edge of the surface can be moved, monolithically, up and down by a small amount.

A good way of thinking about this situation is to imagine that the straight lines drawn on the surface in figure 2 are inextensional *bars*, which are connected in fours at the inner edge, and to the fixed wall at the outer edge, by means of frictionless *joints*. Overall, there are three times as many bars as there are inner joints; and so the simple version of Maxwell's rule (e. g. Calladine 1978), namely

$$b - 3j = 0, \tag{3}$$

where b is the number of bars and j is the number of non-foundation joints, suggests that the assembly is kinematically determinate, i. e. rigid. However, this is evidently not the case; and we are in fact dealing with an assembly which is simultaneously *kinematically and statically indeterminate*, in accordance with the complete form of Maxwell's rule (Calladine 1978), namely

$$b - 3j = s - m, \tag{4}$$

where s is the number of independent states of self-stress and m is the number of independent kinematic mechanisms.

It is not difficult to find a state of self-stress within the assembly, in order to confirm indirectly the kinematic indeterminacy of the assembly: this involves all the inner bars of an indefinitely-long version of the model being in equal tension, with the equilibrium of the joints maintained by tension, of much smaller magnitude, in the nearly-radial bars.

Analysis, Stage 2

The kinematic indeterminacy described above can be seen readily by means of a "displacement diagram", as shown in figure 3(a). Here, each outer triangle —the shaded type in figure 2— is given a small vertical component of displacement w (downwards) at its tip. Since the triangle is hinged to the wall at angle β, and is assumed to remain rigid, there is also a peripheral component of displacement

$$v = w \tan \beta \tag{5}$$

at the tip.

If each of the joints is given precisely the same displacement, then —as shown in figure 3(a)— the length of each inner, inclined bar is not altered; and so the entire assembly can execute a single degree of kinematic freedom while all its members remain inextensional. This is indeed the mechanism exhibited by the simple cardboard model, and described above.

Now all of Guastavino's spiral staircase shells are anchored to rigid structures at the top and bottom of each flight. However, the simple mode described above is plainly incompatible with such supports, which require $w = 0$ at each end.

We should therefore consider a more elaborate pattern of displacement, in which the inner-edge vertical components vary with the polar angle θ, and satisfy the above displacement-conditions at the ends of the sector. In this way the fixity of the ends obliges the shell to deform in a non-inextensional way, thereby absorbing some strain energy. The simplest such mode is

$$w = w_0 \sin (\pi\theta / \lambda) \tag{6}$$

Here (Fig. 2) λ is the angle subtended by the two fixed ends of the shell in plan. In this case, adjacent nodes in the discrete version illustrated in figure 2 will move by different amounts; and in turn this leads to straining of the inneredge bars and, by a straightforward extension to the actual continuous version, straining throughout the spiral surface.

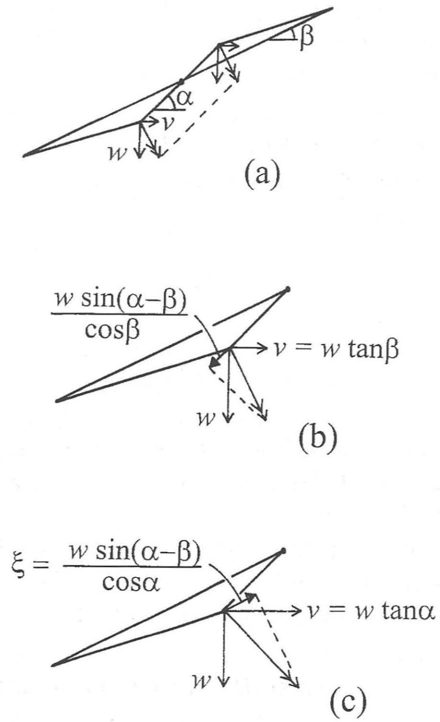


Figure 3
(a) Radial view of an inner triangle, seen edge-on, and two neighbouring outer triangles (cf. figure 2). The double-headed arrows show, greatly exaggerated, small displacements of the inner tips of the outer triangles perpendicular to those triangles, with components w, v (single-headed arrows) in co-ordinate directions r, θ. When, as shown, the displacements of adjacent tips are equal, the inner triangle is evidently not distorted.
(b) Enlarged view of the left-hand part of (a), showing also (solid arrow) the component of displacement of the tip parallel to the edge of the inner triangle.
(c) As (b), but now the displacement of the tip of the outer triangle is perpendicular to the edge of the inner triangle; and the component of the displacement parallel to the edge of the outer triangle is again shown by a solid arrow

Mode I

Consider first an arrangement which we shall call mode I, in which, as before

$$v = w \tan \beta \qquad (5, \text{bis})$$

The near-radial bars remain inextensional, as above, because the displacement of every inner joint is perpendicular to the hinge-line at the outer wall. But now the length of the inner bar can change, if the displacements of the joints at its two ends are different.

We can see from figure 3(b) that the component of displacement of an inner joint in the direction of the inner bar is

$$v \cos \alpha - w \sin \alpha \tag{7}$$

or, in the light of (5)

$$- w \sin (\alpha - \beta) / \cos \beta \tag{8}$$

Here the negative sign indicates that the displacement along the inner bar is in the negative sense of θ: the displacements of the edge nodes are everywhere in a "downhill" sense over the sector.

If two consecutive joints, at either end of an inner bar initially subtending angle $\Delta\theta$ in plan at the center, have vertical displacements w and $w + \Delta w$, respectively, then the inclined bar has an extension, by (8), of

$$- \Delta w \sin (\alpha - \beta) / \cos \beta, \tag{9}$$

where the negative sign indicates a shortening.

Now this bar is originally of length

$$a\Delta\theta \sec \alpha, \tag{10}$$

and so it experiences a strain

$$\varepsilon = - \frac{\Delta w}{a\Delta\theta} \frac{\sin (\alpha - \beta) \cos \alpha}{\cos \beta} \tag{11}$$

Thus, for the small displacement prescribed by (5), (6) we have —on taking the limit $\Delta w / \Delta\theta \to dw / d\theta$—

$$\varepsilon = - \pi w_0 A \cos (\pi\theta/\lambda) / \lambda a \tag{12}$$

where the trigonometrical function $A(\alpha, \beta)$ is given by

$$A = \sin (\alpha - \beta) \cos \alpha / \cos \beta \tag{13}$$

Thus, in this mode the inner bars are shortened at the bottom of the shell, and lengthened at the top.

So far we have evaluated the strain ε in the inner bars for the case where the near-radial bars are all inextensional. If we now revert to the original continuum, and imagine that straight, radial lines within the original surface remain straight —a reasonable kinematic first hypothesis— we have a strain-field in which there is no shearing strain $\gamma_{r\theta}$, while the circumferential strain ε_θ decreases in magnitude from ε as in (12) at the inner edge $r = a$ to zero at the outer edge, $r = b$. This variation is not far from linear, but its deviation from true linearity depends on the geometric ratio b/a.

In the strain-energy analysis to be performed later on, we shall need to integrate ε_θ^2 over the surface area of the staircase. It will be most convenient to do this by first evaluating the mean value of ε_θ^2 over the plan area of the staircase; and when we do this we find

$$\varepsilon_\theta^2 \Big|_{\text{mean}} = \frac{1}{2}\left(\frac{1}{3} - \frac{k}{12}\right)\left(\frac{\pi w_0}{a\lambda}\right)^2 A^2 \tag{14}$$

where

$$k = (b/a) - 1 \tag{15}$$

In (14) the multiplier (1/2) is the mean value of $\cos^2(\pi\theta/\lambda)$ over the range $0 \le \theta \le \lambda$, while the factor $((1/3) - (k/12))$ is the mean value in the radial direction: it would be 1/3 for an effectively "straight" staircase with (b/a) only marginally greater than 1; and the reduction by $(k/12)$ gives a fair approximation to an integral which can be tedious to evaluate exactly. Since we shall need to evaluate $\varepsilon_\theta{}^2$ over the actual area of the helicoid, as distinct from the area in plan, yet another geometrical factor will be required. For the sake of simplicity, and without introducing an error of more than a few percent into a calculation which is already not exact, we shall take

$$\text{Area of helicoidal sector / Area in plan} = \sec((\alpha + \beta)/2); \tag{16}$$

i. e. we shall take the "average slope" of the surface as a whole as the mean of α and β.

Mode II

So far we have described a mode of small deformation of the helicoidal sector in which the vertical displacement of the inner edge is given by (6), while the interior distortion of the surface involves only circumferential strain, with mean-

value given by (14). That mode comes about as a direct consequence of the prescription (5).

Another simple possibility would be to take

$$w = w_0 \sin(\pi\theta/\lambda), \tag{6, bis}$$

as before, but now to prescribe

$$v = w \tan \alpha \tag{17}$$

instead of (5).

In this case the component of displacement of the tip of the triangle parallel to the inner bar is zero; and so mode II, as we shall call it, leaves the inner bars unchanged in length.

It is now the turn of the near-radial bars to alter their lengths somewhat. To investigate this, let us calculate the component of displacement of an inner node parallel to the outer edge of the shaded triangle of figure 2, i. e. at inclination β to the horizontal. Calling this component of displacement ξ, we have, from figure 3(c),

$$\xi = v \cos \beta - w \sin \beta;$$

hence, by (17)

$$\xi = w B \tag{19}$$

where

$$B = \sin(\alpha - \beta) / \cos \alpha \tag{20}$$

Here, as far as the outer triangle of figure 2 is concerned, there is a small "uphill" displacement of the tip, which is directly proportional to w.

In terms of the continuous surface, we thus find an in-plane *shear distortion*, which may be characterized by a "nominal" shear strain

$$\gamma_{nom} = \xi / \lambda \tag{21}$$

$$= (w_0 / l) \sin(\pi\theta / \lambda) B \tag{22}$$

where

$$l = b - a \tag{23}$$

In evaluating the elastic strain energy of distortion for this mode we shall need to compute the integral of over the surface of the helicoidal sector. We could do this, as in mode I, by assuming that straight radial lines in the undeformed surface remain straight during this distortion; but it is simpler to proceed by analogy with a plane annular elastic disc which is held at its outer edge while the inner edge is given a uniform peripheral displacement ξ. In this way we find the mean value

$$\gamma^2_{r\theta}\Big|_{\text{mean}} = \frac{1}{2}\, C^2\, \frac{(Bw_0)^2}{l^2} \tag{24}$$

where

$$C = (1 + k) \,/\, (1 + k/2) \tag{25}$$

Here, essentially as in (14), the factor (1/2) represents the mean value of $\sin^2(\pi\theta/\lambda)$ around the circumference of the sector, and the term involving k accounts for the non-uniformity of the shear-strain with radius. For this mode, radial lines in the original surface do not remain precisely straight during the deformation; but the departure from straightness is small.

Expression (24) gives, approximately, the mean of $\gamma^2_{r\theta}$ over the plan area of the sector, and so the extra factor (16) will again be required in order to evaluate the strain energy of shearing over the helicoidal surface.

Energy calculations

As already mentioned, we shall investigate the structural action of the helicoidal sector by balancing the internal elastic strain energy with the work done by the external loads during the assumed modal displacement.

We shall assume that the loading force is vertical and downwards, of intensity f per unit area. Further, we shall assume that f is independent of r, but that it varies sinusoidally with θ in the same manner as w. Thus we take

$$f = f_0 \sin(\pi\theta/\lambda) \tag{26}$$

Our aim, of course, will be to find a "stiffness" or "flexibility" relationship between f_0 and w_0; and if we can do this, we shall be able to investigate other distributions of f around the circumference by means of Fourier superposition. (In the present paper, however, we shall be concerned only with the fundamental Fourier term (26)).

In line with our previous calculations, we need the mean value of the intensity of the work done by the vertical load on the vertical displacement of the helicoidal surface as the modal deflection is increased from zero to w_0. In this way we find, by integration over the plan area:

$$\text{external work}\Big|_{\text{mean}} = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}+\frac{k}{6}\right) f_0 w_0 \tag{27}$$

Here the first factor (1/2) corresponds to the fact that force is directly proportional to displacement; the second factor (1/2) is the mean value of $\sin^2(\pi\theta/\lambda)$ around the circumference of the sector; and the factor involving k accounts for the trapezoidal shape of a radial slice.

In the calculation leading to (27) it is assumed that the load intensity is uniform with radius, whereas the vertical component of displacement varies linearly from the inner radius to zero at the outer radius. Later on we shall consider briefly the frequency of natural vibration of the shell, for which purpose the load, by d'Alembert's principle, varies with radius in the same way as displacement. For this case the term containing k in (27) must be replaced by

$$\left(\frac{1}{3}+\frac{k}{12}\right), \tag{28}$$

as may readily be shown.

Modal flexibility

We are now in a position to estimate the modal flexibilities of modes I and II, respectively, by equating the strain energy of distortion in each mode to the work done by the loads.

For mode I we shall assume that the strain energy per unit area of surface is equal to

$$Et\varepsilon_\theta^{\,2} / 2, \tag{29}$$

where E is the Young modulus of elasticity of the material and t is the thickness of the shell. Implicit here is an assumption that Poisson's ratio is zero, which is justifiable on the grounds of simplicity in this preliminary investigation: in any case the value of Poisson's ratio for ceramic tiles is small.

As we have already evaluated the mean values of the necessary quantities, the energy-balance for mode I is:

$$(Et / 2)\varepsilon_\theta^2 \Big|_{\mathrm{mean}} \sec((\alpha + \beta) / 2) = (1 / 4)(1 / 2 + k / 6) f_0 w_0 \tag{30}$$

Hence, using (14) and tidying up:

$$\left.\frac{w_0}{f_0}\right|_{\mathrm{I}} = \frac{(1/2 + k/6)}{(1/3 - k/12)} \frac{\cos((\alpha + \beta)/2)}{Et} \left(\frac{a\lambda}{\pi A}\right)^2 \tag{31}$$

The flexibility of mode I thus turns out to be directly proportional to λ^2, other things all being equal.

In a similar way we find for mode II:

$$\left.\frac{w_0}{f_0}\right|_{\mathrm{II}} = \frac{(1/2 + k/6)}{C^2} \left(\frac{k}{B}\right)^2 \frac{\cos((\alpha + \beta)/2)}{Gt} a^2 \tag{32}$$

Here, G is the elastic shear modulus of the material. The flexibility of mode II is thus independent of λ.

A plot of $w_0 / f_0|_{\mathrm{II}}$ against β, according to (32), is given in figure 4(a) for several values of b/a. For this purpose it is convenient to define a dimensionless factor Φ, so that

$$\left.\frac{w_0}{f_0}\right|_{\mathrm{II}} = \Phi \frac{a^2}{Et} \tag{33}$$

Although mode II involves shearing of the shell's surface, and hence the elastic modulus G of the material, it is convenient to express G in terms of Young's elastic modulus E. For present purposes it is reasonable to take

$$G = E/2 \tag{34}$$

which is equivalent to taking Poisson's ratio again as zero. We shall use (34) for the remainder of this paper.

Factor Φ may be written as the product of two factors, which are functions of k and α, β respectively:

$$\Phi = \phi_1(k) \cdot \phi_2(\alpha, \beta), \tag{35}$$

where

$$\phi_1 = k^2 (1 + k/3) (1 + k/2)^2 / (1 + k)^2 \tag{36}$$

and

$$\phi_2 = \cos((\alpha + \beta)/2) \cos^2\alpha / \sin^2(\alpha - \beta) \tag{37}$$

It turns out (see figure 4 (a) that Φ varies with β as β^{-2}, approximately; and that Φ is rather insensitive to the value of *b*/*a*.

In some first-order calculations that we shall perform below, we shall take 3 as a "typical" value of Φ, noting that a practical range of this variable might lie between 6 and 1.5.

How can we decide, for a staircase of given parameters —namely *a*, *b*, β, λ, *t*, *E*— which of our two competing modes is the more appropriate? Now the energy method which we have used gives, in general, an *over-estimate* of the stiffness of a given mode (e. g. Calladine 1983, Appendix 1). Thus the better of the modes I and II in a given case is the one which gives the lower stiffness, i. e. the higher value of flexibility factor w_0 / f_0.

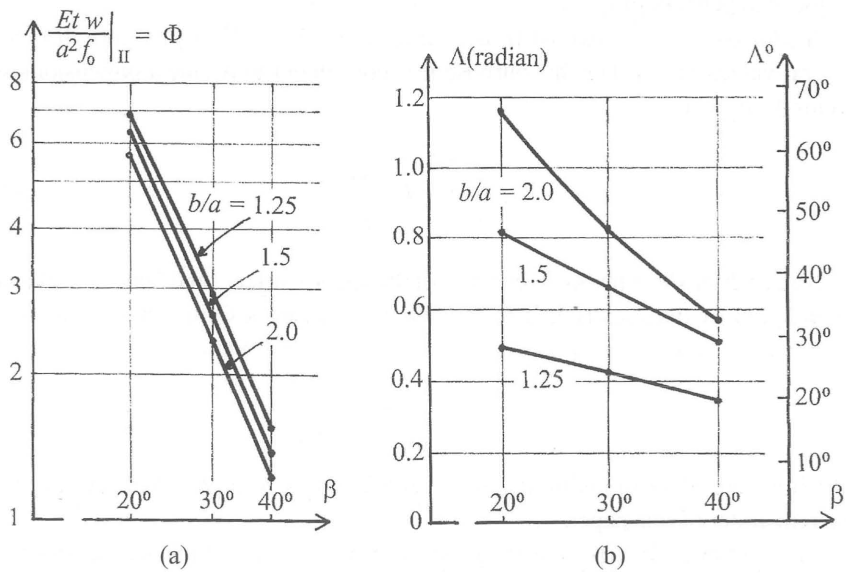


Figure 4
(a) Logarithmic, dimensionless plot of w_0 / f_0 for mode II against outer climb-angle β, for a practical range of β. The flexibility is, roughly, inversely proportional to the square of β, and independent of *b*/*a*.
(b) The "crossover" value, Λ of λ, plotted against β for three values of *b*/*a*. Λ is given in radians on the left and degrees on the right

Now it is clear, since w_0 / f_0 is proportional to λ^2 for mode I, but is independent of λ for mode II, that mode I is to be preferred for high values of λ, while mode II is better for low values of λ. And in order to give substance to the meaning of "high" and "low" in this context, we should evaluate what we may call the "cross-over" value of λ, say

$$\lambda_{\text{crossover}} = \Lambda, \tag{38}$$

for which (31) and (32) coincide. Thus we obtain, after tidying up:

$$\Lambda = \pi \left(\frac{2}{3}\right)^{1/2} \frac{k(1 - k / 4)^{1/2}\,(1+k / 2)}{(1 + k)} \frac{\cos^2\alpha}{\cos\beta} \tag{39}$$

Figure 4(b) shows a plot of Λ against β for three practical values of b/a. We see in general that Λ decreases as β increases and as b/a decreases. We might take $\Lambda = 40^\circ$ as a typical value; and since the value of λ for a typical staircase flight might be between 90° and 180°, we can see that deflection in mode I will usually be prominent.

The relationship between the flexibilities of the two modes may now be written simply as

$$\left.\frac{w_0}{f_0}\right|_{\mathrm{I}} = \left(\frac{\lambda}{\Lambda}\right)^2 \left.\frac{w_0}{f_0}\right|_{\mathrm{II}} \tag{40}$$

Modes I and II are not mutually exclusive, of course. Indeed, we should expect to get a more realistic result by using a "combined" mode of the kind

$$w = w_0 \sin (\pi\theta / \lambda) \tag{6, bis}$$

$$v = w\left(\zeta \tan \beta + (1 - \zeta) \tan \alpha\right) \tag{41}$$

In other words we may superpose ζ times mode I and $(1 - \zeta)$ times mode II, where ζ is a variable.

For a given set of design parameters we should now seek the value of ζ which *maximizes* the value of (w_0 / f_0), in order to make the best use of our two modes.

The calculation is straightforward, and the result is, simply

$$\zeta = 1 / \left(1 + (\Lambda / \lambda)^2\right); \tag{42}$$

so that, finally,

$$\frac{w_0}{f_0} = \left.\frac{w_0}{f_0}\right|_{\mathrm{I}} + \left.\frac{w_0}{f_0}\right|_{\mathrm{II}} \tag{43}$$

$$= \left(1 + \left(\frac{\lambda}{\Lambda}\right)^2\right) \left.\frac{w_0}{f_0}\right|_{\mathrm{II}} \tag{44}$$

In other words, the optimum combination of the two modes results in the straightforward *addition* of the two flexibilities (31) and (32).

A direct consequence of this result is that the distribution of shearing strain within the surface turns out to be directly proportional to f_0, irrespective of the value of λ. Thus, as we have seen in (21), from the analysis of mode II, the shearing strain in the spiral surface is directly proportional to $w_0|_{\mathrm{II}}$, other things being equal. In other words, if f_0 has been specified, the shearing strain is determined, irrespective of the value of λ and the contribution from mode I towards the total of w_0. By contrast, the distribution of *circumferential strain* ε_θ within the surface is proportional to the product $f_0\lambda^2$.

Why should the shear strain and the circumferential strain depend in such different ways on the angle λ? To answer this question we first note that these strains are directly related to the corresponding stress components by the respective elastic moduli. Although we have been working in kinematic quantities, rather than in stress quantities which satisfy the equations of statical equilibrium, we may nevertheless argue that by minimising the stiffness f_0 / w_0 with respect to ζ we are coming closest to satisfying overall the conditions of equilibrium, subject of course to the strong constraint of our assumption that the deformation of the surface under load can be decomposed into the two simple modes which we have chosen.

Membrane equilibrium equations

The most direct way of thinking about the membrane-equilibrium of a thin shell under purely vertical loading is to use Pucher's transformation (e. g. Timoshenko and Woinowsky-Krieger 1959, secc. 113), in which the various stress-resultants in the shell, namely the radial and circumferential direct-stress resultants N_r, N_θ, and the shear-stress resultant $N_{r\theta}$, are "mapped" onto their counterparts $\overline{N}_r$, $\overline{N}_\theta$, $\overline{N}_{r\theta}$, in a corresponding horizontal, *plane* problem. The applied vertical forces are then carried by the sum of each of these stress-resultants multiplied by its corresponding component of curvature or twist; meanwhile, the conditions of equilibrium in the horizontal plane are precisely those of the plane problem, and may be satisfied by the use of an Airy stress-function.

When we apply Pucher's idea to the present problem, two simplifying factors appear. First, the curvature of the helicoidal surface is zero in both radial and circumferential directions; so that vertical external loads must be carried directly by the product of $N_{r\theta}$ and the *twist* of the surface. Second, the quantity $\bar{N}_{r\theta}$, in the companion plane problem, which corresponds to $N_{r\theta}$ in the helicoidal surface, is actually *equal* to $N_{r\theta}$.

From this point of view, then, it should not be surprising that the shear strain $\gamma_{r\theta}$ is directly related to the applied loading f_0, irrespective of the value of λ.

But why should ε_θ, and by implication N_θ, turn out to be proportional to λ^2, so that the overall deformation under a given loading f_0 is dominated by ε_θ when $\lambda > \Lambda$, even though N_θ apparently contributes nothing directly to equilibrating the imposed loading?

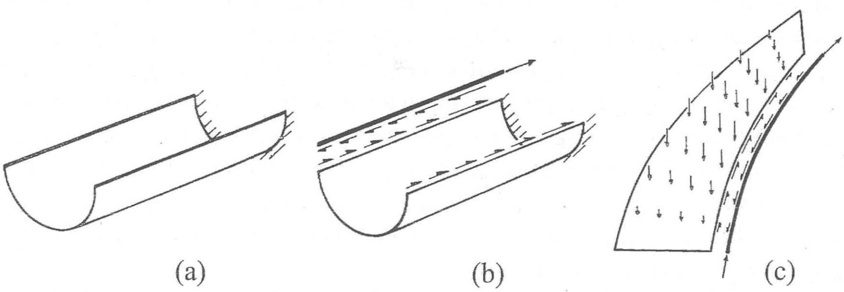
To answer this question we must appreciate that stress-resultants derived directly by satisfying the equations of equilibrium, according to the "membrane hypothesis" —which may well have a unique solution, as mentioned earlier— sometimes require "strings" of "concentrated tension" along their edges (e. g. Timoshenko and Woinowsky-Krieger 1959, sec. 105; Calladine 1983, sec. 4.2, 4.4). A straightforward example is a semi-circular inverted barrel-shell as a simply-supported cantilever-beam and carrying its own weight, as shown in figure 5 (Calladine 1983, section 4.4). Throughout the shell the longitudinal stress-resultants as calculated by the equations of equilibrium according to the membrane hypothesis are *compressive*; and so the overall longitudinal equilibrium at any cross-section of the shell can be satisfied only if there is a "tendon" along each free edge. The membrane-equilibrium analysis thus tells the designer that some sort of edge-thickening or prestressing tendon is required: otherwise the shell will distort under load in such a way as to set up stresses which do indeed satisfy the requirements of longitudinal equilibrium. If the shell is sufficiently *thick*, of course, it will behave as a beam according to the classical "engineers'" theory of cantilever beams; and the extent to which that happens will depend on the value of an appropriate dimensionless group.

In general, the tension in an "edge tendon" of this kind, as required by the strict application of the membrane-equilibrium equations to the shell proper, will vary along its length in accordance with the shear-stress which is demanded by the membrane-equilibrium equations at the free edge. This is shown schematically in figure 5 (b) for the case of the barrel-shell.

There is of course a similar situation along the free inner edge of our hyperboloidal staircase-shell. There we have postulated, for mode II, a shearing strain —and hence by implication a shearing-stress-resultant $N_{r\theta}$— which has a finite value at the inner edge, $r = a$. Since $N_{r\theta}$ is proportional to $\sin(\pi\theta/\lambda)$, the corresponding tendon-tension must vary as $\cos(\pi\theta/\lambda)$ —apart from a possible additive

constant— being compressive at the bottom of the staircase and tensile at the top. This is shown schematically in figure 5 (c) —although the situation is not directly analogous to that of 5 (b) since the tendon is curved, and other forces (not shown) are required for equilibrium. The variation of ε in mode I is precisely of this kind; and we must suppose that somehow the structural effect of such a "tendon" is "smeared out" over a finite width of the staircase shell by some kind of "boundary-layer" phenomenon, which we shall not investigate further here. It is clear that the maximum tension/compression in such a "tendon" is directly proportional to λ; and it follows from an energy balance, analogous to the one which we have already performed, that the overall contribution of circumferential straining to the deflection of the shell must be proportional to λ^2, other things being equal.

A further aspect of a "statical" interpretation of our kinematic/energy-balance analysis of the staircase shell is the *distribution* of $\gamma_{r\theta}$ in the radial direction. By using the analogy of an annular disc held at its outer edge and rotated peripherally at its inner edge, in order to evaluate the strain energy of shear distortion, we are in effect assuming that $\gamma_{r\theta}$ varies as r^{-2}. It also turns out that the *twist* of the helicoidal surface varies as r^{-2}. Hence the vertical load intensity that would be carried, as in Pucher's analysis, by the product of $N_{r\theta}$ and twist varies as r^{-4}, instead of being *uniform*, as we have supposed.



(a) (b) (c)

Figure 5

(a) Cantilevered half-cylinder shell carrying its own weight: the "membrane-equilibrium" solution requires "tendons" along the straight edges (Calladine 1983, figure 4.8).

(b) As (a) but showing that the membrane-equilibrium equations require unbalanced shearing stresses along the straight edges of the shell; which are provided by reactions from the edge-tendons, one of which is shown detached.

(c) The mode II solution for the staircase shell loaded in accordance with (26) implies unbalanced shearing stresses along the free edge: the view is that of figure 2. These can be provided by reactions from an edge-tendon, here shown detached, as in (b) —but with some forces omitted, that are required on account of the curvature of the tendon

The difference between these two distributions suggests that the mode-forms which we have chosen for our approximate "energy" analysis are too crude to model, even approximately, some significant details of the static equilibrium of elements within the shell. But we may still be confident that our crude modes I and II provide *under-estimates* of the shell's flexibility, by virtue of the energy theorem which we have used. The simplest way of improving the quality of the energy-method calculations would probably be to make our simple modes I and II rather more elaborate by assuming a non-linear variation of w with r. But that would make the working a good deal heavier, of course.

Another, more practical, way forward would be to perform, by means of commercially-available computer software, a range of finite-element computations for shells having different geometrical parameters. It would be particularly interesting to investigate the effect of increasing λ, other things being equal — particularly if the ratio E/G could be varied, artificially, so as effectively either to suppress mode I or to enhance its role.

Possible contribution from bending stiffness

Throughout the paper we have assumed that the structural action of the shell involves primarily "membrane" stresses and strains. Here we shall make an order-of-magnitude assessment of the possibility that flexural effects might also be important.

Suppose we were to make a series of radial cuts, so that the shell surface became a series of cantilevers of length l projecting inwards from the rigid outer wall. Under a uniform pressure-loading of intensity f_0, the tip deflection w_0 of such a cantilever is equal to

$$f_0 l^4 / 8EI \tag{45}$$

where I is the second moment of area of a beam of unit width and thickness t. This gives

$$w_0 / f_0 = 1.5 l^4 / Et^3 \tag{46}$$

Now according to figure 4 (a) a typical value of the flexibility of the shell in mode II might be

$$\left.\frac{w_0}{f_0}\right|_{\mathrm{II}} = \frac{3a^2}{Et} \tag{47}$$

Thus the flexibility of the cantilever exceeds that of the shell by a factor

$$l^4 / 2a^2t^2 \tag{48}$$

Typically this will be of order 100 for a practical shell; and so we may conclude that the shell will indeed carry most of the load by means of membrane action.

That calculation is, of course, very crude. Thus, we have ignored the trapezium shape of the cantilever; the constant in (47) may be higher or lower by a factor of 2; and so on. The conclusion would be less convincing, of course, if the shell had $\lambda/\Lambda > 1$, for then the shell's flexibility might be considerably higher, according to (44).

Static deflection of a staircase-shell under load

An engineer would like to know by how much a particular staircase will deflect when it is loaded with people. Formulas (33), (39) and (44) above may be used for this purpose, once the parameters of the staircase have been fixed. However, it is possible to investigate the order-of-magnitude of the deflection of typical staircases by inserting "typical" values into the formulas.

Let us suppose, initially, that our shell is known to deform primarily according to mode II.

For a shell of thickness 0.1 m —say 3 layers of tiles— and a material of mass-density

$$\rho = 2{,}000 \text{ kg/m}^3 \tag{49}$$

—again a typical value— the superficial density of the shell is

$$200 \text{ kg/m}^2 \tag{50}$$

Taking the mass of a typical person as 70 kg, we see that the mass of the shell is equivalent to the mass of about 3 people per square metre. A heavily-loaded staircase might support 6 people per square metre; so we could take a vertical distribution of force of type (26), with

$$f_0 = 4{,}000 \text{ N/m}^2 \tag{51}$$

as a typical live-load.

From (33) we have

$$w_0\big|_{II} = \Phi a^2 f_0 / Et \tag{52}$$

Taking

$$E = 15 \times 10^9 \text{ N/m}^2 \tag{53}$$

as a typical value for tile-and-mortar, 3 as a typical value of Φ (as above), and

$$a = 1.5 \text{ m} \tag{54}$$

as a typical inner radius, we find

$$w_0\big|_{II} = 0{,}02 \text{ mm} \tag{55}$$

Now the total static deflection under load depends, of course, on the ratio λ/Λ, according to (44).

From figure 4 (b), a "typical" value of Λ might be 40°, and an upper limit on λ is about 180°. In such a case $\lambda/\Lambda = 4.5$, so that total deflection is about 20 times the deflection for mode II; which gives a final deflection of about

$$w_0 = 0.4 \text{ mm} \tag{56}$$

This small order-of-magnitude is a direct consequence of the fact that the applied loading is carried by in-plane *membrane* stress.

Frequency of natural vibration

It is a straightforward matter to adapt our modal-stiffness calculation to an approximate computation of the natural frequency of vibration of the shell.

If we assume a mode of the form

$$w = w_0 \sin(\pi\theta / \lambda) \cos \omega t, \tag{57}$$

where ω is the circular frequency, equal to $2\pi n$, where n is the frequency measured in Hz, then by d'Alembert's principle we have an inertia loading

$$\begin{aligned} f &= \rho t \ddot{w} \\ &= \rho t \omega^2 w_0 \sin(\pi\theta / \lambda) \cos \omega t \\ &= f_0 \sin(\pi\theta / \lambda) \cos \omega t \end{aligned} \tag{58}$$

Here, as above, ρ is the density of the material; superior dots denote differentiation with respect to time; and the symbol t when in the combination ωt denotes time.

Hence,

$$\omega^2 = \frac{1}{\rho t}\frac{f_0}{w_0}, \tag{59}$$

where w_0 / f_0 is the previously computed modal flexibility. However, as noted already, if we wish to have a force distribution in the radial direction which matches the displacement distribution —i. e. varying linearly with r to zero at $r = b$— then the factor $(1/2 + k/6)$ in formula (27) must be replaced by $(1/3 + k/12)$.

In this way we obtain, for vibration in mode II,

$$n = \frac{1}{2\pi a}\left(\frac{1/2 + k/6}{1/3 + k/12}\right)^{0.5}\left(\frac{E}{\rho}\right)^{0.5}\left(\frac{1}{\Phi}\right)^{0.5} \tag{60}$$

Note that this expression is independent of the thickness of the shell. Taking the typical values quoted above for E and ρ, we find:

$$(E / \rho)^{0.5} = 2{,}700\text{m/s}. \tag{61}$$

Taking now 3 as a typical value of Φ (see figure 4 (a)), 1.5 m as a typical value of a, and 1 as a typical value of k (i. e. $b/a = 2$), we get

$$n \approx 210 \text{ Hz} \tag{62}$$

This corresponds to a note about one-third of an octave below middle C. Going to the highest/lowest value of Φ in figure 4 (a) would lower/raise the frequency by about half an octave. This calculation, however, overlooks several factors, as follows.

First, we have assumed in calculating the d'Alembert inertia loading that the displacement is w, whereas its amplitude is really $(w^2 + v^2)^{0.5}$. This can make a significant difference, particularly in mode II, where $v = w \tan \alpha$; but it can be allowed for when the mode is known. Second, we have done the calculation for mode II alone, whereas there may be a significant, or even dominant, contribution from mode I if —as will normally be the case— λ exceeds Λ. This factor may also be included, once the dimensions of the shell have been fixed. Third, we have taken the mass as the mass of the shell by itself, i.e. not including the mass of the added steps. The mass of the steps might well be roughly equal to the mass of the shell; in which case the frequency would be reduced by a further half-octave.

Discussion

I have presented here a preliminary, exploratory structural analysis of a generic helicoidal timbrel staircase shell; and I am conscious that I have left over a number of loose ends. I hope that others will be able to test the accuracy or otherwise of my predictions of natural frequency and deflection by working numerically with some actual examples of Guastavino's spiral staircases, and testing the validity of my parameters Φ and Λ (Fig. 4).

It may be useful for me to close by giving a summary of the main points in the analysis as it has developed.

First, I argued that the *membrane* theory of thin elastic shells is the appropriate tool for analysing the Guastavino "cohesive", "timbrel" thin-shell construction. Then I suggested that a kinematic approach, using elastic energy theorems with assumed modes of small deflection, would be more advantageous initially than an approach *via* statical equilibrium —particularly for making a first estimate of deflections under load and frequencies of natural vibration of timbrel shells.

For an extended (i. e. endless) helicoidal shell there is a simple —and indeed disconcerting— mode of inextensional deformation; and this must be countered by fixing the upper and lower ends of the sector. When the angle λ subtended in plan by the sector is sufficiently small, the shell deforms primarily by means of in-plane shearing strain; but when λ is large, circumferential stretching strains become significant. (This seems to be broadly analogous to the relationship between deflection of a loaded beam on account of shear and flexural straining, respectively, as a function of the length of the beam). These two modes of deformation (modes II and I, respectively) produce overall deflection under load of the kind

$$K_1 + \lambda^2 K_2 \tag{63}$$

where K_1, K_2 are constants, other things being equal. It is convenient to determine the value, Λ, of λ at which these two contributions are equal, in terms of the geometrical parameters of the staircase. The ratio λ/Λ is then an important indicator of the mode of action of a particular staircase.

The overall flexibility of the staircase under load is always inversely proportional to the thickness t of the shell, other things being equal; and Λ is independent of t. The fundamental natural frequency of vibration of the shell itself is also independent of thickness, since stiffness and mass are each proportional to t; but that calculation takes no account of the mass of the steps which are added in order to convert the spiral shell into an actual staircase.

A simple, order-of-magnitude calculation confirms that bending effects add a generally negligible contribution to the stiffness provided by in-plane membrane action within the shell.

Throughout the paper I have assumed that the primary structural action is provided by the thin, helicoidal shell of tiles and mortar, built Catalan-style. This is quite different from the mode of action of staircases built from individual stone steps which are keyed into the supporting wall and rest on one another's edges. The structural action of such staircases has been elucidated by Professor Heyman (1995) through rigid-body statics. Whether or not a completed "Guastavino" spiral staircase carries some of its imposed load by means of analogous action depends upon the way in which the monolithic masonry treads are keyed into the supporting wall —a matter on which I have no information.

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The mechanical sciences in Antiquity

Massimo Corradi

The study of the mechanical sciences (Duhem 1903, 1905; Dugas 1950; Benvenuto 1981), in spite of having its roots in antiquity, underwent a slow evolution in the centuries that preceded the Renaissance, and then progressed ever more rapidly, up to the time when, at the end of the nineteenth century (Marcolongo 1904), the fundaments of rational mechanics were established. On the other hand, the development of applied mechanics made reasonable progress from the Greeks throughout the whole of the Middle Ages, to then begin, starting in the Renaissance (Galileo 1638) a "machine revolution" that led to an acknowledgment and formal development until it achieved, in the nineteenth century, the renewed linguistic expression of the theories which, as time went on, were supported by experimental discoveries and were interpreted on the basis of theoretical principles (see, for example, the 1588 text of Agostino Ramelli (1531–1591) and the 1629 text of Giovanni Branca (1571–1645) (Biral and Morachiello 1985; Marchis 1994; Micheli 1995). The progress of mechanical concepts, however, came about in a different way with respect to the evolution of the history of constructive technologies; in fact, if in the latter case it is possible to see a linear progress that grew through time, in the former we are dealing with a recent acquisition that can be traced by to the beginning of the twentieth century, with the appearance of constructions in cast iron, wrought iron, steel, reinforced concrete and, finally, of "new" materials such as aluminium, high-resistance steels, epoxy resins, reinforced fibreglass and composites, of which today ample use is made as a "panacea" for resolving problems related to the conservation of ancient and monumental structures.

If Aristotle (384–322 B.C.) and Archimedes (287–212 B.C.) set forth the principles that are at the basis of mechanics, the former in the *Φυσικη ακροασιζ* and in the *Μηχανικα Προβληματα* (attributed to Aristotle by Cardinal Bessarione (Micheli 1995, 133–136; Micheli 1996, 29–37; Bottecchia 1982), and the latter with the *De aequiponderantibus*, in like manner the development of elementary constructive systems —such as the trilith system, the flat arch and the arch— followed different routes, tied more to the world of technologies than to that of science (Di Pasquale 1996). Thus the problems faced by architects and engineers were resolved through the use of those very mechanical principles that lie at the base of the discipline but without a precise knowledge of their general semantic value; rather with an awareness that was both static and constructive, an empiric knowledge that validated the choices made on the building site, with surprising results from the constructive and formal points of view.

The fundaments of the discipline

The principles that lie at the basis of mechanics can be traced back —in the spirit of Jean-Baptiste Le Rond d'Alembert (1717–1783) (D'Alembert 1743; 1758)— to a *reduction ad unum*, that is, it is possible to trace everything back to a unique principle, one which is the fundament of the discipline and which is by itself capable of expressing the foundations of mechanics, statics and kinematics: the *principle of virtual work*.

As Edoardo Benvenuto so capably demonstrated (Benvenuto 1981: 3–14), this principle is a descendant of causality, of the Aristotlean idea of a supernatural Being, by which all that moves is moved by something, that is, by an immobile First Cause that is at the origin of universal motion. From the power of the First Cause that moves all descends the *principle of virtual displacement*, the *principle of virtual velocity*, and finally, the *principle of virtual work*; this leads —implicitly— to the equilibrium of a rigid body by means of the unique equilibrium equation $\boldsymbol{M} = \boldsymbol{0}$, implying the fact of the equilibrium of the resultant vector of the system of forces ($\boldsymbol{R} = \boldsymbol{0}$). Such a concept represents, in the broader context that unites philosophy and theology, the antithesis of a monotheistic idea conceived in the context of a polytheistic religion, the figural extension of the unique Legislator that governs the world. What is being dealt with in this case is the precognition of a theological principle which sees in the one God the supernatural Being that governs Nature, the destiny of the world, and in consequence, that of men; all that creates or is created is part of a larger design of which it is given to us to know only the material event, what has happened or will happen, but not the reason why, with no possibility of reaching the original cause. All of this represents

a reunification of Science and Faith which was, for many centuries, and perhaps today as well, denied by religion and by science itself.

The *principle of virtual work*, deduced from the equilibrium of Aristotle's lever, aptly expresses —as Benvenuto led us to observe in his 1981 essay— that perfect dichotomy between mechanics and geometry, which may be explicated in terms of force or displacement, so as to assume a role that goes beyond the merely scientific aspect of the antithesis between statics and kinematics, but which resides in the more general vision of Christian thought, in which all things that are created, both "visible and invisible" (S. Paul, *Letter to the Colossians* 1:16) has a single origin. Relative to this see the *querelle* of the nineteenth century in the context of rational mechanics relative to the debate of whether statics precedes kinematics or vice versa. In like manner, by virtue of this very principle, the two disciplines, of different speculative origins, summarise in a single formula the contraposition that exists between the world of statics and that of kinematics, between algebra and geometry, between numbers and vectors, in an alternation of entities that can be defined axiomatically and represented geometrically, in which one substitutes the others and all in their turn contribute to explain the nature of the physical world around us.

On the basis of such a premise, it can be confirmed with complete certainty that the theoretical fundaments of mechanics can be traced by to the Aristotle's *Φυσικη*, but traces of it are also found in classic Greek philosophy. The term *Μηχανη* —intelligence, deceit, artifice (*Iliad* VIII, 177)— or even the intelligence related to machines, sheds light on the contraposition of the mechanical arts, which appertain to the reign of astuteness, and therefore of all that is against nature, and physics, which is occupied with what happens according to nature. Mechanics regards the construction of artificial entities, of *instruments* aimed at bending to their own uses the forces of nature: how is it possible that "the lever, with a small effort, permits the lifting of great weights?" (Krafft 1970).

In any case, if in Aristotle we find implicitly the principles formulated on the basis of speculation and in the form of concepts that are purely philosophical and abstract, mechanics demonstrates a more explicit connection with geometry in the writings of Archimedes (Napolitani 1998), whose ideas —as was the case with Aristotle as well— will come down through the centuries of the development of the discipline, until Galileo (1564–1642) in the seventeenth century would establish the "order" of the *New Science* in his *Discourses and Mathematical Demonstrations Relating to Two New Sciences* of 1638.

If it was Aristotle who took on the task of founding physics as a science, the physics that he founded is far from modern physics, which bends all to the language of mathematics: Aristotle, on the contrary, deals with a qualitative physics (here too, as we shall see, in contrast to Plato), which not rarely tends to become

metaphysics. Indeed, in Book VI of *Metaphysics* the Stagirite proposes his classifications of the sciences, and does not only present physics as a fully-fledged science, but indeed inserts it in the category of "theoretical sciences" (together with mathematics and metaphysics), which, having as their ends knowledge itself, are disinterested, hence their superiority. Instead, the practical sciences (ethics and politics) have an action as their end, while the poietical sciences aim at the production of objects. Indeed, in *Metaphysics* VI, 1025 b 25, Aristotle writes:

> Therefore, if all thought is either practical or productive or theoretical, physics must be a theoretical science, but it will theorize about such being as admits of being moved, and about substance-as-defined for the most part only as not separable from matter. Now, we must not fail to notice the mode of being of the essence and of its definition, for, without this, inquiry is but idle. … If then all natural things are a analogous to the snub in their nature; e.g. nose, eye, face, flesh, bone, and, in general, animal; leaf, root, bark, and, in general, plant (for none of these can be defined without reference to movement-they always have matter), it is clear how we must seek and define the 'what' in the case of natural objects, and also that it belongs to the student of nature to study even soul in a certain sense, i.e., so much of it as is not independent of matter. That physics, then, is a theoretical science, is plain from these considerations.

For this reason the Stagirite searched in metaphysics to find the first principles of being, defining the notions of substance (*ουσια*) and accident (a determination inherent in substance), potential and act. Thus becoming is explained as a passage from one body in a state of potentiality to that in action, subsequent to the action of something that is already in action. In order to avoid recalling the infinite, a concept that was among other things *in fieri*, Aristotle postulates as the origin and cause of motion the existence of an immobile First Cause, eternal, incorruptible and always acting, from contact with which movement is generated. In Aristotelian logic language as a means of knowledge permits distinguishing between terms, substances and attributes; besides, the examination of propositions (the union of substance and attribute) permits the establishment of formal conditions for the validity of the reasoning and their requisites for truth (see the doctrine of syllogism). However, physics —in contrast to metaphysics, which studies being as being— is a particular science, because it is concerned only with being in motion and for that reason is deficient in universality. In this sense, the node of the Stagirite's treatment of physics concerns theology: does nature tend to an end, or rather do all things take place according to the bizarre rules of chance? To this question Plato responded by playing the car of providential finalism, hypothesising the existence of a "Divine Artificer" (the demiurge of the *Timaeus*) so as to contemplate eternal ideas and to connect them to the material

in the best possible way: the world that was thus derived was the best of all possible worlds, a kind of infallible artwork in which all was sustained by divine strings. Aristotle's solution was very different indeed: Platonic providentiality is set aside, because nature is not at all divine and Aristotle compares its activity to technical ones. In contrast to divine technology (which is infallible), human technology is subject to failure; similarly, in nature not everything occurs perfectly. Nature can commit errors, just as man can, although its actions are always and in any case oriented towards the best. The possibility of error on the part of nature is made possible by the fact that it takes place in a context of material things; matter cannot always be dominated by forms, and sometimes poses resistance. Thus, beginning with these considerations, the Stagirite has to examine the factor of "casuality", and does so in chapters 4, 5 and 6, arriving at the conclusion that the single case falls into the category of the cause, meaning that even that which happens by chance has a cause, even if the effects turn out to be the result of a different cause than the usual one, that is, it happens "by accident". Concerned with the physics of bodies in motion, Aristotle lingers a great while on the idea of movement (the whole of book III is dedicated to that topic), and recognises four possible modes of motion (*κινησιζ*), local, substantial, quantitative and qualitative, definitively dismantling the Eleatism of Parmenides.

Μηχανικα Προβληματα

The 1613 print edition of the *Μηχανικα Προβληματα* of Francesco Maurolico (1494–1575) contains the Aristotelian *Quaestiones Mechanicarum*, divided into 35 queries and an *Appendix nonnullarum Quaestiones Mechanicarum per Maurolycum* comprising forty-six problems and the addition of an additional thirteen problems taken on by the mathematician from Messina (Maurolico 2005). Maurolico's objective is very ambitious, in that he wants to present Aristotle's *Quaestiones Mechanicarum* from a different scientific and philosophical perspective, particularly the topics that are most closely related to statics, convinced that he has a new key for reading and interpreting the ideas of the Greek philosopher. It is a fact that the work of the Stagirite, which was almost completely unknown for the whole of the Middle Ages, was rediscovered in the 1500s (Aristotle 1497; Vittore 1517), becoming a point of reference for all discussions of mechanical problems (Moody and Clagett 1960; Rose and Stillman 1971; Clagett 1972).

Maurolico undertakes a new reading of Aristotelian thought by means of the Archimedean mechanics in his *De aequiponderantibus*, better known as *De momentis aequalibus*, the work of Maurolico that deals with Archimedes's text (Maurolico 1685). He gives particular attention to the concepts of "centre of

gravity" and "moment", translating, in fact, notions that are substantially qualitative (Galluzzi 1979, 74–89) which express the action of the weight on the lever and the balance, that is, the medieval "science of weights", expressed by the more general concept of *gravitas secundum situm*, into a *momentum* expressed quantitatively and defined by a very precise axiomatic system. In fact, the Messinese mathematician does not rigorously define *momentum* but describes it by means of the law of equilibrium of the lever (Maurolico 1685, I: def. 8–112, post. 3–5), a kind of Ariadn's thread that allows him to see the equilibrium equation as a rotation of a system of parallel forces. In this sense, speaking of forces, Maurolico extrapolates the concept of *impetus*, which he considers to be one of the four fundamental magnitudes of mechanics, along with *corpus*, *pondus* and *momentum*, retracing mechanics to an intermediate discipline between mathematics and physics; in any case, very often in his writings there transpires a more traditional interpretation of mechanics as an intermediate science between arts and technologies, going back to the question of the dichotomy of *Ars* and *Τεχνη*, that is, between science and technology. The same references to a literature that moves between Vitruvius's *De architectura*, Aristotle's *Problemata* and Heron's *Spiritalia* demonstrates the broadness of the horizons of thought of this mathematician from Messina.

The principles set forth by Archimedes —even while re-presenting Aristotelian thought, and in particular the problem of the lever related to the *principle of virtual work*— open, by means of geometry, into the world of mechanics, of instruments, of mechanical devices, as illustrated by Plutarch in his *Vita di Marcello* (Plutarch 2000). The mathematician from Syracuse, beginning with a basis in theory, uses the fundaments of geometry and mechanics for the construction of machines with practical ends, as related by Plutarch when he tells of the siege of Syracuse.

> These machines he had designed and contrived, not as matters of any importance, but as mere amusements in geometry; in compliance with King Hiero's desire and request, some little time before, that he should reduce to practice some part of his admirable speculation in science, and by accommodating the theoretic truth to sensation and ordinary use, bring it more within the appreciation of the people in general. Eudoxus and Archytas had been the first originators of this far-famed and highly prized art of mechanics, which they employed as an elegant illustration of geometrical truths, and as means of sustaining experimentally, to the satisfaction of the senses, conclusions too intricate for proof by words and diagrams (Plutarch, *Marcellus*).

Eudoxus's idea was harshly criticised by Plato, who maintained that the mathematician from Cnidos abandoned abstract and intelligible principles that

were the fruit of theoretical speculation in order to pass to the "sensible", to material bodies, separating —in fact— mechanics from geometry, transferring mechanics to the sphere of *Τεχνη*, and thus removing it from philosophy. A *reductio in terminis*, in certain ways, from a speculative science to a practical science, or rather from to an art at the service of war.

In contrast, in Archimedes thought are present concepts that appertain to the world of statics, but that also have their "geometric evidence", that is, the *principle of symmetry* and the *principle of sufficient reason*. Even if such principles were not directly pronounced by Archimedes, but extrapolated only in the fifteenth century by medieval masters of mechanics, they are obvious for what they are in the Archimedean postulates relative to the equilibrium of the level and the balance. They are the precursors to the studies of Pierre Varignon (1654–1722) and Leonhard Euler (1707–1783) on the equilibrium of rigid bodies (1740) and on the equilibrium of a system of forces (1697) (cardinal equations of statics, formulated by Euler in his 1749 treatise entitled *Scientia navalis seu tractatus de construendis ac dirigendis navibus*).

Even if Archimedes's writings have come down to us in an incomplete form (see, for example *L'equilibrio dei piani*, *Sui galleggianti*, *La quadratura della parabola*, "ex traditione Maurolyci" (*Opera Archimedis ex traditione Maurolyci*, *Demonstrationes quaedam de centro gravitatis*, *Fragmenta*, 1685), they left a significant mark on medieval mechanics, to which they were passed down through the studies of Archimedes by Pappus of Alexandria (fourth century A. D.). Indeed, the mathematician from Alexandria cites the Archimedean *Μηχανικον* and *Σαιροποιια* as the texts that earned Archimedes the name of *meccanico per eccellenza*.

Archimedean mechanics —which takes on statics, kinematics, hydrostatics, and the construction of machines— as well as the applied mechanics of Heron of Alexandria (first century B. C.) gave rise in the Middle Ages to the development of simple machines, uniting *μηχανηματα* and geometry, complementary disciplines in which mechanical principles can be represented through the use of Euclidean geometry and elementary algebra. The definition of simple machines is due to the fact that these concretised mechanical concepts that could be immediately comprehended. The graphic representation of such machines appears in the texts with the birth of the printing press beginning from the fifteenth century; they translate into images the descriptions of such *μηχανηματα* as are found in the Arab translations of ancient texts, certainly with many interpretative and representational distortions, having undergone several linguistic transpositions. This explains the reason why machines called by the same name appear in different forms or why the same name indicates instruments that have different constructive origins or different fields of application.

The concepts present in the *Φυσικη* of Aristotle are in like manner taken up again and rewritten by Galileo in his *Mechanics*, a manuscript of 1593, and in the *Discourses* previously cited, published in Leiden by Louis Elzevir.

As was noted by Alexandre Koyré in 1976, Archimedean mechanics and Aristotelian thought come together in Galilean science and in the foundation of a mechanics and science of construction (Galileo 1638) that are deductive and abstract, where experience is the fundamental tool for the authentication of a theory, but where the experiment is performed by Galileo only through the reasoning of Euclidean geometrical space and with the use of the simple laws of algebra and the theory of proportions. The *sensate sperienza* of Galileo introduces the idea of experimental proof: indeed, the Pisan scientist underlined more than once how theory that conflicts with the data of an experiment had to be refused because "even a single contrast with experience is decisive proof of the falsity of the theory", because what gives sense to the experience is its immediate mathematical translation (Corradi 2003). With Galileo it is possible to perceive in all its force how the task of the mechanic is not that of stupifying ("one commands nature only by obeying") but of bending oneself to the laws of nature and benefiting from knowledge of them. Mechanics is no longer only practical knowledge, a non-liberal art, a collection of empirical procedures for the construction of machines, automatons or instruments, but has become a "rational" science, that is, one that conforms to the laws of nature and is equal in stature to the exact sciences.

It appear clear however that the development of the mechanical sciences in the seventeenth century does not take place within a consolidated discipline but is the fruit of the development of the philosophical thought which from Plato (428/427–347 B. C.) to Archimedes, from Eudoxus (390–340 B. C.) to Archytas of Tarentum (428–347 B. C.) to Euclid (ca. 325–ca. 270 B. C.) has intertwined natural philosophy and the mathematical sciences, that is, mechanics and geometry. As Giuseppe Cambiano writes, "the reduction of geometry to mechanics was equivalent to inverting the Platonic itinerary from geometry to dialectics" (Cambiano 1996).

In this context, the observations of Plutarch (46–120 A. D.) are significant; referring to Archimedes, he speaks of *οργανικη* in the lateral sense of the machine and not of a purely theoretical mechanics. Further, Plato's condemnation of Eudoxus and Archytas —held by him to be guilty of having reduced geometry to the *οργανικη* and of having neglected the theoretical basis in favour of the practical applications of mechanics, without having traced them back to any fundamental principle— revaluates Achimedean thought: here Plutarch distinguishes the principles that are useful for the comprehension of the philosophical from the

nature that had pervaded all of Greek thought up to the threshold of the Christian era, identifying in philosophic thought, in mathematics, and in consequence, in geometry, the tools that permit the translation of mechanical principles into the construction of machines.

Indeed, when Archimedes wrote *On the Equilibrium of Planes* and the treatise *On Floating Bodies*, the developed the problems of statics and hydrostatics by means of geometry, introducing concepts such as the centre of suspension or the centre of gravity, the equilibrium of bodies, hydrostatic thrust, etc., uniting the principle of the machine, *ratio* to *fabrica*, equivalent to performing an inductive-deductive process of scientific thought which would be developed successively in the Renaissance, as for example in the *Liber mechanicorum* of Guidobaldo del Monte (1545–1607) (Del Monte 1577), in which the theory of simple machines is set forth rationally and thoroughly.

The *Mechanicae artes*, or in other words the construction of mechanical instruments and of military machines —among which was the *sambuca* ideated by Heraclides and used by Marcellus in the siege of Syracuse according to Plutarch (Plutarch 2000)— showed the development of mechanics applied to machines that was perfected in the construction of war machines, hydraulic and pneumatic machines, automatons, by a historic process of perfecting empirical processes that are acquired and passed down through time: that is, they give rise to what is commonly called today "technology" (Cambiano 1991). The *γαστράετειζ* (crossbow), torsion or pulley trebuchet, ballista, catapult, etc., show how a discipline that is taking its first steps towards its formation from a theoretical point of view is perfectly capable of being immediately translated in machines, devices, instruments, that is, in practical applications, since empirically it had already made its own the basic principles of the discipline. The fundaments of the discipline —of Aristotelian and Archimedean origins— will find in the nineteenth century their theoretical connotations in rational mechanics and in mechanics applied to the construction of machines. With regards to this, it is appropriate to recall that almost all the editions of Vitruvius's *Ten Books on Architecture* (first century B. C.) published in the Renaissance first in Latin and then in the vulgar and finally translated into all languages, give ample space to machines, be they machines for building, or mechanical instruments for raising water, military machines, etc. (Vitruvius 1521, 1536, 1584). Given this, it is possible to understand how this combination of knowledge (mathematic, mechanic and constructive) formed, in the luminous years of Greek, Roman, medieval and Renaissance architecture, the cultural toolbox that was proper to the architect, in that the architect was the artificer of the developments which, successively, would become mechanics applied to construction. The problem of knowledge, according to Plato, was in fact closely related to the problem of the communication of knowledge.

Arguing with the Sophists, Plato held that ideas are the basis of being and knowing, ideas in the sense of pure and intelligible essences that reside in the Hyperuranium, ordered hierarchically in the supreme idea of the Good. Thus, to the gnoseological dualism expressed in the contraposition of opinion and science corresponds the metaphysical dualism between the sensible world of becoming and the immutable world of ideas, knowledge of which is *anamnesis* or *reminiscence*. On the basis of this principle is founded the doctrine, of Pythagoraean-Orphic origin, of the immortality of the soul (expressed in the *Phaedo*) and, on the principle of dualism between the sensible and the intelligible worlds, the doctorine of *eros* (love), that is, the desire for knowledge that mediates dialectically between the senses and reason, knowledge and the practical. Plato rejected the possibility of a natural science, because he was convinced that this would be subject to the incessant flow discussed by Cratilus, and he maintained that there might be no *επιστημη* (science, knowledge, cognition) of the natural world but rather *δοξα* (credence). The *Timaeus* itself, which was also a dialogue dedicated entirely to*'υσιζ* (nature), is configured as a tale that is rich in imagery but with basis whatsoever in fact.

In the contraposition of ideas and the terrestrial world, between theory and empiricism, with a predilection for the former, as explained in the fable of the cave, which symbolises the genres of sensible and supersensible, imagination and credence, true being and Ideas, is found the archetype, present as well in the history of science, of the dualism of science and technology where in any case the study of theory is given priority, relegating to the background the investigation of constructive technology in the empiric sense of the term, demonstrated by the fact that this discipline developed at the beginning of the twentieth century. The mystic and theological aspects of Platonism are evident in this vision of the world: life in the reign of the senses and the sensible is the life of the cave, just as the life of the reign of the spirit is life in pure light; the moving from sensible to intelligible is expressly represented as a conversion; and the supreme vision of the sun and of the light itself is the vision of the Good and contemplation of the Divine. Thus the formative dialogue is based on knowing how to reason, interpret and resolve, that is, the path of knowledge where *sensible knowledge and intellective knowledge are dialectically interwoven and neither one nor the other, by itself, can insure the true knowledge of the meaning of things*.

In conclusion: simple machines

Medieval mechanics explicated its principles of mechanics through the use of simple machines: the lever, scale, pulley, inclined plane, wedge, screw. This

amounts to a reduction of the principle through the knowledge of instruments that are easy to apply, capable however of demonstrating with immediacy evidence of useful if not indispensable mechanical phenomena for the construction of not only machines but, in the words of Galileo, *fabbricar navilii, palazzi o templi vastissimi* "building ships, palaces or the most vast temples" (Galileo 1638), that is, the art of construction and of architecture in general.

In like manner, architecture developed beginning with the elementary static systems, such as trilith, flat arch, pseudo-arch and pseudo-vault, arch, vault and dome, in an ordered succession of structural elements that went from the simple to the complex, from a system of weights to a system of thrusts, from light systems to elastic systems, using only elementary static principles capable of explicating and conjoining the behaviours of systems of weights and forces, of actions and reactions (*principle of symmetry*, *principle of sufficient reason*, *principle of action and reaction*, etc.).

The mechanical principles present in the Greek mechanics of Aristotle and Archimedes —of which the simple machines are none other than exemplifications by means of easy-to-understand interpretative instruments— are at the basis both of construction in the strict sense and of architecture in a lateral sense, both of the machines used for hoisting and the placing of materials. The common

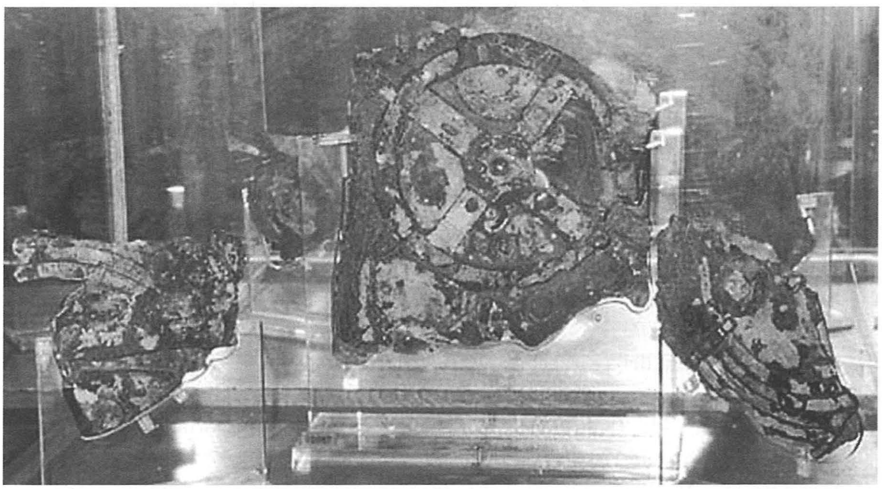

Figure 1
Remains of the Antikythera Mechanism in the Archaeological Museum of Athens, an astronomical instrument probable used to determine the relationships between the sun, moon, earth and stars

point of departure is the equilibrium of the lever, from which descend the *principle of symmetry* and the *principle of sufficient reason*, which together with the *principle of virtual work*, implicitly formulated by the Stagirite and better exemplified by Archimedes, are the basis for the development of architecture. In an iter than covers some twenty centuries of history, mechanics goes from the studies of Herone (first century B.C.) to Pappus of Alexandria (third-fourth centuries A.D.), Jordanus Nemorarius (thirteenth century), Leonardo da Vinci (1452–1519), and regards the study of resistance and traction of cables, the bending of beams, the conditions for equilibrium of bodies by means of the "polygon of support", on the statics of arches, finally reaching Varignon's *Nouvelle Méchanique* at the end of the seventeenth century. But this history is well known and can be found in numerous books on the history of science and technology for those who wish to know more.

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On the safety of the masonry arch. Different formulations from the history of structural mechanics

Federico Foce

A decalogue of false hypotheses

In 1927 American engineer George Fillmore Swain published a noteworthy treatise of *Structural Engineering* whose third volume, devoted to *Stresses, graphical statics and masonry*, contains a whole chapter on the stone arch that deserves great attention. In the footnote at the beginning of the chapter Swain advises the reader with these words:

> Since the stone arch is an elastic arch, differing only in degree from a monolithic concrete arch, it is impossible to distinguish sharply between the two. Before the student of structural engineering begins the study of elastic arches, it is desirable that he should study carefully this chapter on the stone arch, notwithstanding the fact that stone voussoir arches are now seldom built (Swain 1927, 400).

This warning is meaningful. Behind it we read an old attitude reflecting the distinction, also in terms of mechanical behaviour, between *constructions en maçonnerie* and *constructions en charpente,* a *Leitmotiv* of the technical literature on the *science des ingénieurs* since its beginning. At the date of publication of Swain's treatise this attitude was "officially" banished from the studies of structural engineering for a very obvious reason: it was unanimously accepted that stone, as well as steel or wood, are characterized by the common properties of strength and elasticity, even though with different degree and specific features. Thus, the elastic methods used for steel and wood structures can be reasonably applied to stone and masonry constructions.

This is exactly why Swain's words are of special interest. He was a distinguished scholar of elasticity and strength of materials and, as a young student, he had spent some time in Berlin studying theory of elastic systems under Emil Winkler, a pioneer in the field of the *Elastizitätslehre* and one of the first authors to support the application of the elastic analysis to the stone arch (Winkler 1879). Despite this scientific training, Swain's view stands out from the chorus and probably represents the best methodological lesson from a professional elastician. He writes:

> the elastic theory seems to be firmly intrenched in American engineering literature. Perhaps some who use it do not realize its defects and assumptions, and like it because it is complex and mathematical. It seems to be a curious characteristic of the human mind that it so often prefers complexity to simplicity, and mistakes obscurity for profundity . . . The writer believes in elastic methods, if they are necessary; not if they are unnecessary and if a simpler method is just as good (Swain 1927, 424).

This explicit position against any acritical application of the elastic theory to the masonry arch is accompanied by a decalogue listing the theoretical hypotheses assumed in the elastic analysis and, in parentheses, a brief remark on the real state of things:

The elastic theory is often termed "exact". The assumptions made in it are the following.

1. That the ends are rigid and do not rotate (this is untrue).
2. That the span does not change at all (this is untrue).
3. That the material is homogeneous (this is untrue).
4. That the modulus of elasticity is constant, not changing with the pressure (this is untrue, though perhaps close).
5. That the terms with *r* in the denominator may be neglected (this may be far from true).
6. That the integrals may be replaced by summations (this is approximate).
7. That the formulae for flexure are exact (this is untrue).
8. The stresses due shrinkage are neglected.
9. That the section is a rectangle (this is untrue; see Art. 18).
10. That the loads may be determined accurately (this is untrue; both the loads and their distribution on the arch are quite uncertain).

Possibly to some minds, not too mathematical, these facts may justify some of the conclusions in the present chapter (Swain 1927, 425).

These conclusions, and the "simpler method" that Swain refers to in the previous quotation, can be summarized with the following arguments, based on a particular use of Winkler's theorem on the minimum of the deformation work. As known, this theorem states that for an arch of constant section, under vertical loads, the true line of thrust is approximately the one which lies nearest the axis of the arch ring, in the sense that the sum of the squares of the vertical deviations is a minimum. Starting from this theorem, Swain observes that if we can draw within the arch ring the lines of minimum and maximum thrust,

> it is reasonably certain that there is a line nearer the center line than either of these, and that the arch is stable. The writer therefore believes the statement to be true that *if any line of resistance* (that is a line of thrust) *can be drawn within the arch ring the arch is stable*; and if a line of resistance can be drawn within the middle third, the true one will also be within the middle third, and there will be compression over the whole of every joint (Swain 1927, 412).

Swain is perfectly aware that this way of applying Winkler's theorem does not give the true line of thrust. To locate it, it should be necessary to find the line that actually minimizes the sum of the squares of the vertical deviations, as Baker remarked polemically. But, Swain replies,

> it is not necessary to do all this. It is only necessary to observe that, starting with the maximum line and gradually reducing the thrust and raising the point of application in the crown, the line will gradually change from the maximum to the minimum line, and that surely there will be some line that will be nearer the center line than either maximum or minimum, in order to conclude that if any line can be drawn in the arch which is not at the same time maximum and minimum, the true line will be in the arch and the arch is stable (Swain 1927, 414).

Obviously, Swain continues,

> if it were necessary to find accurately the true line, it would be necessary to do what Professor Baker says; and of course the *stresses* at the edges of a joint cannot be found accurately unless the true line is found. But this is unnecessary to judge *stability*. The stresses can be computed with quite sufficient accuracy without doing all this. The writer considers it a useless expenditure of time to try to find the true line anyway, considering the many uncertainties of the problem, regarding loads, their distribution, the material and workmanship, and even the so-called accurate elastic theory itself (Swain 1927, 414)

To conclude, Swain suggests that the stone arch, as well as the plain or reinforced concrete arch, can be studied with complete reliance on the results, if the followings are admitted:

1. The true line of resistance is one lying nearest the center line.
2. As we pass gradually from the maximum to the minimum line, some line is sure to be found which is nearer the center line than either the so-called maximum or the minimum.
3. It is not necessary to compute the stresses at the edges of the joints with extreme exactness (Swain 1927, 424).

It is easy to understand that these three points and of Swain's arguments appear somewhat familiar if we look at them from the viewpoint of modern limit analysis. Points 1 and 2 state, on the whole, that if we can draw within the arch the lines of maximum and minimum thrust, then the arch is stable because, in accordance with Winkler's theorem, the true line will also be within the arch. Point 3 clearly asserts that we do not need to find the true line of thrust, because the actual stresses cannot be computed with exactness. In other words, local strength is of secondary importance if global stability is ascertained. Even though Swain's deductive reasoning is based on Winkler's theorem, which is a theorem of elasticity, his conclusions are perfectly correct. From our modern point of view, he gives the right answer to what Jacques Heyman, after the plastic "revolution" occurred in the theory of structures in the 1950s, was to call "Poleni's problem" in one of his most convincing paper on the methodology to be assumed in the case of the masonry arch (Heyman 1988). As a matter of fact Swain's attitude, despite his elastic-oriented education, is totally in tune with the old tradition of studies which Heyman himself first brought to light with his pioneering work on the limit analysis of the stone skeleton (Heyman 1966).

There is no need here to quote Heyman's well-known theoretical studies on the matter. It is sufficient to say that the mark left by these studies was indeed profound. Proof of this is found, on one hand, by the constant references made to them in the technical literature, and on the other hand, by the numerous translations and reproductions of his works in recent years. But this is not all. To Heyman goes the credit for having promoted a new research methodology that is still today largely outside the mainstream of the interests of the structural engineer. We are speaking of that line of research aiming at a critical reconstruction of the historical development of the structural disciplines by rereading the sources and comparing them with ideas, methods and knowledge of the present time, the same line of historical research pursued from the 1960s by Clifford Truesdell in the context of the mechanics of solids and materials and later followed by Edoar-

do Benvenuto and Salvatore di Pasquale in the field of structural mechanics. It is to these masters that the present paper intends to render homage.

Basic features for the collapse analysis of the masonry arch

Before discussing in some detail the principal eighteenth- and nineteenth-century contributions on the theory of the stone arch, we give here a brief presentation of the stability and collapse conditions of the arch modelled in accordance with Heyman's three well known hypotheses concerning masonry behaviour, that is: 1) masonry has no tensile strength; 2) masonry has an infinite compressive strength; 3) sliding failure cannot occur.

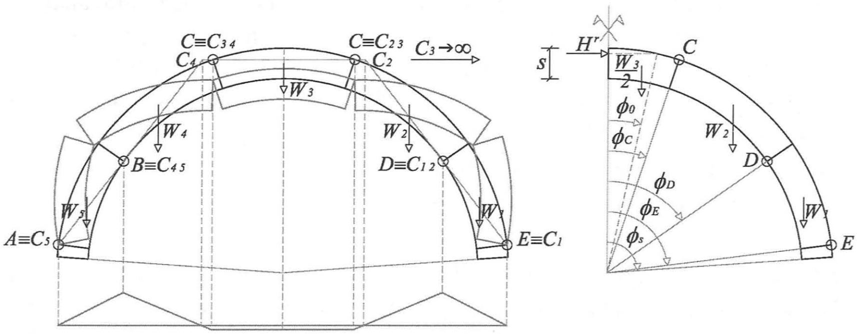

Figure 1
Collapse Mode I in its general form and corresponding kinematical chain

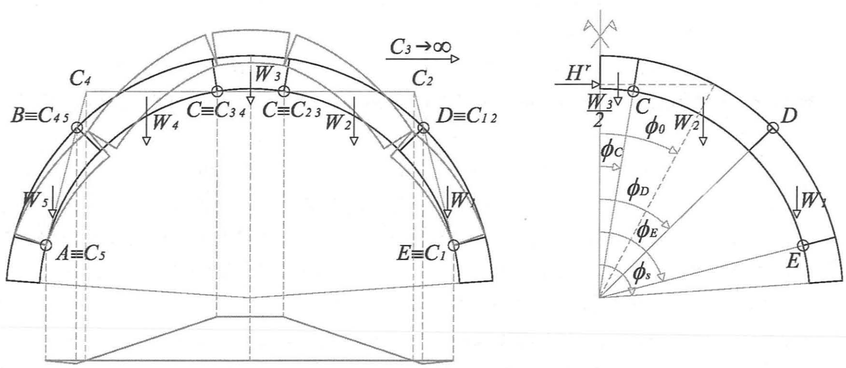

Figure 2
Collapse Mode II in its general form and corresponding kinematical chai

For the sake of simplicity, let us consider a symmetric arch of constant thickness s subject to a symmetric load. Two opposite rotational collapse modes with one degree of freedom may occur. Their general form is shown in figures 1 and 2 with the corresponding kinematic chain.

The angle ϕ_0 has been introduced in order to define the application point of the thrust H at the crown joint. In particular, if $\phi_0 = \phi_C = 0$ we find the two "usual" modes with hinge at the extrados or at the intrados of the crown joint, respectively (Figs. 3 and 4).

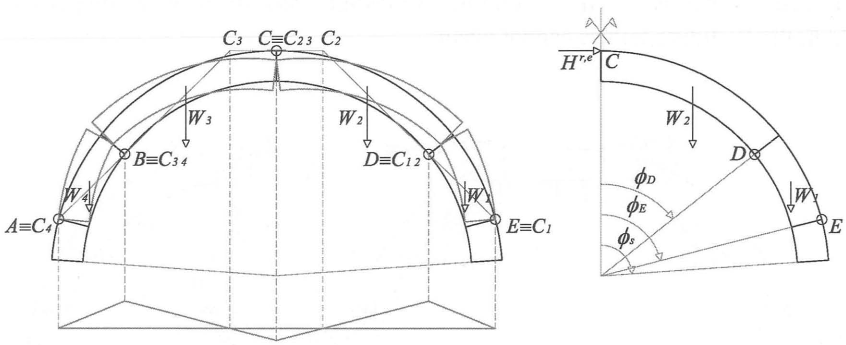

Figure 3
Collapse Mode I in the case of hinge at the crown extrados

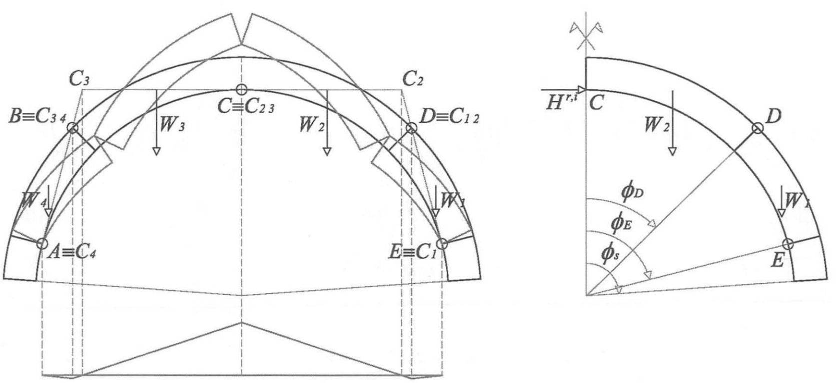

Figure 4
Collapse Mode II in the case of hinge at the crown intrados

The collapse analysis can be developed in terms of equilibrium equations or in terms of principle of virtual work. Whatever the approach, the collapse condition is the only one which is, at once, both statically and kinematically admissible with respect to a chosen collapse parameter, for instance the thickness of the arch.

Collapse analysis in terms of equilibrium equations

Let us consider a symmetric arch under a symmetric load and call $H^r_{\min}(\phi, \phi_0, s)$ and $H^r_{\max}(\phi, \phi_0, s)$ the values of horizontal thrust applied at a generic point of the crown for the equilibrium of a half arch about the intrados M and the extrados N of the joint at angle ϕ, respectively (Fig. 5). Given ϕ_0 and s, the first is a minimum, the latter a maximum.

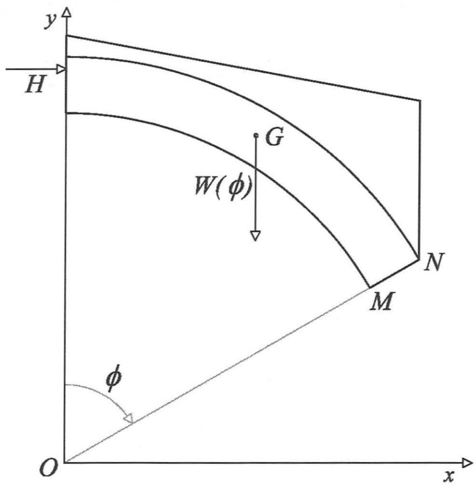

Figure 5

In order to avoid rotation about the intrados edge of any joint of the arch it must be

$$H \geq \max H^r_{\min} \tag{1}$$

In order to avoid rotation about the extrados edge of any joint of the arch it must be

$$H \leq \min H^r_{\max} \tag{2}$$

so that the necessary and sufficient condition for the equilibrium of the arch is

$$\max H^r_{\min} \le H \le \min H^r_{\max} \tag{3}$$

The necessary condition of collapse then becomes

$$\max H^r_{\min} = \min H^r_{\max} \text{ (statically admissible thrust)} \tag{4}$$

where $\max H^r_{\min} = H^r_{\min}(\phi_D) =$ and $\min H^r_{\max} = H^r_{\max}(\phi_C) = H^r_{\max}(\phi_E)$ for Mode I and $\min H^r_{\max} = H^r_{\max}(\phi_D) =$ and $\max H^r_{\min} = H^r_{\min}(\phi_C) = H^r_{\min}(\phi_E)$ for Mode II.

Condition (4) is also sufficient if the angles ϕ_C, ϕ_D and ϕ_E satisfy the disequalities

$$\phi_C < \phi_D < \phi_E \text{ (kinematically admissible mechanism).} \tag{5}$$

For thrust applied at the crown extrados (Mode I for $\phi_0 = \phi_C = 0$) (5) and (6) become

$$\max H^{r,e}_{\min} = \min H^{r,e}_{\max} \tag{7}$$

and

$$0 < \phi_D < \phi_E \tag{8}$$

where $\max H^{r,e}_{\min} = H^{r,e}_{\min}(\phi_D)$ and $\min H^{r,e}_{\max} = H^{r,e}_{\max}(\phi_E)$. Similarly, for thrust applied at the crown intrados (Mode II for $\phi_0 = \phi_C = 0$), (5) and (6) become

$$\max H^{r,i}_{\min} = \min H^{r,i}_{\max} \tag{9}$$

and

$$0 < \phi_D < \phi_E \tag{10}$$

where $\max H^{r,i}_{\min} = H^{r,i}_{\min}(\phi_E)$ and $\min H^{r,i}_{\max} = H^{r,i}_{\max}(\phi_D)$.

The previous analysis obviously admits a derivation in terms of line of thrust as well. Let us define first in a general way the properties of the so-called lines of minimum and maximum thrust for a symmetrical arch.

The line of minimum thrust is the steepest one possible within the arch ring, i. e., the most extended vertically and contracted horizontally; it necessarily touches the extrados at two symmetric points *e* near the crown (or at the extrados of the crown) and the intrados at two symmetric points *i* near the springings (or at the springings) (Fig. 6 a);

The line of maximum thrust is the flattest one possible within the arch ring, i. e., the most contracted vertically and extended horizontally; it necessarily touches the intrados at two symmetric points *i* near the crown (or at the intrados of the crown) and the extrados at two symmetric points *e* near the springings (usually at the springings) (Fig. 6 b);

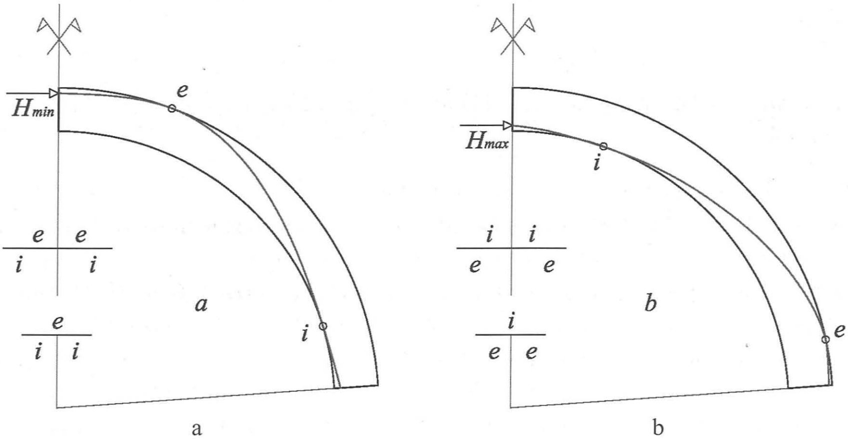

Figure 6
General form of the lines of minimum and maximum thrust for a symmetric arch

In terms of line of thrust, the collapse condition states that the arch fails only if the the lines of maximum and minimum thrust coincide, that is if only one line is possible and fulfils the condition for both maximum and minimum thrust.

Collapse analysis in terms of principle of virtual work

For rotational collapse modes the virtual work involves only the actives forces and is a function of the type $\delta L^{(a)}\ (\phi_D, \phi_D, \phi_D, s)$. Obviously, the work function depends on the type of collapse mode, so that a function $\delta L_I^{(a)}$ for Mode I and a function $\delta L_{II}^{(a)}$ for Mode II must be defined. If, for a given thickness and for any compatible mechanisms, it is

$$\delta L^{(a)} < 0 \tag{11}$$

then the equilibrium is stable and collapse cannot occur. The collapse condition requires

$$\delta L^{(a)} = 0 \tag{12}$$

so that the collapse thickness can be determined by means of the two following theorems:

Static theorem: Among the set of the statically admissible states, the collapse thickness is the minimum thickness for which a kinematically admissible mechanism exists;

Kinematic theorem: Among the set of the kinematically admissible mechanisms, the collapse thickness is the maximum thickness for which a statically admissible state exists

Discussion of the principal pre-elastic historical theories on the collapse of the arch

In the section that follows a selection of historical works on the collapse of the arch is critically discussed and compared with the previous results. This selection inevitably neglects other important studies for which we refer to the bibliographies contained in recent works on the matter (Foce 2002, Kurrer 2002, Huerta 2004). However, it collects the main contributions to a general formulation of the collapse analysis of the arch from the point of view of the modern limit analysis based on Heyman's hypotheses. In this sense, our review of the historical sources will be exclusively focused on the rotational collapse modes, even though some of them take into account also sliding collapse modes in the presence of friction.

Coulomb (1773)

Coulomb's analysis in terms of method of maxima and minima is probably the first attempt at a general formulation of the collapse of symmetric arches. Supposing the thrust to be applied at a generic point of the crown and considering the rotational equilibrium of a voussoir, Coulomb correctly finds the necessary and sufficient condition of equilibrium (3). In the *Remarque I* of his *Essai* he adds that the horizontal thrust must act at the crown extrados "pour rendre la force B_1 [max H^r_{min} with our notations] aussi petite qu'elle puisse être" (Coulomb 1776, 380), so that he takes max $H^r_{\text{min}} \equiv$ max $H^{r,e}_{\text{min}}$. Nothing is said, however, about the application point of the thrust in order to compute the other extreme value min H^r_{max} and, consequently, to define the range of admissible thrusts. Two interpretations can be given of Coulomb's text: 1) he implicitly understands that the min H^r_{max} must be found with the thrust applied again at the crown extrados: in this case it should be min $H^r_{\text{max}} \equiv$ min $H^{r,e}_{\text{max}}$; 2) he implicitly supposes that the min H^r_{max} must be found with the thrust applied at the crown intrados, in order to take the greatest value of min H^r_{max}: in this case it should be min $H^r_{\text{max}} \equiv$ min $H^{r,i}_{\text{max}}$.

According to the first interpretation, Coulomb would give only (7), disregarding (8). The analysis of the rotational modes is correct as far as Mode I, but not

complete in that Mode II is not considered. According to the second interpretation, which is the one usually assumed by the exegets of Coulomb's *Essai*, the equilibrium condition would be

$$\max H^{r,e}_{\min} \leq H \leq \min H^{r,i}_{\max} \tag{13}$$

If the range of thrust shrinks to a single value we have

$$\max H^{r,e}_{\min} = \min H^{r,i}_{\max} \tag{14}$$

Now, this equality cannot be the necessary condition of collapse because the two extreme values of the thrust are computed taking two different points of application at the crown joint. No rotational mode can occur, unless the crown joint is reduced to a single point, because only in this circumstance there would be no distinction between extrados and intrados. In this case the analysis of the rotational modes is incorrect, as pointed out by Persy (1825).

Mascheroni (1785)

Lorenzo Mascheroni is one of the few authors to have dealt with the equilibrium of the arch in terms of principle of virtual work. Before him a significant application of this principle to structural problems was given in 1743 by the "tre mattematici" in the known *Parere* on the stability of Saint Peter's dome. During the nineteenth century this approach was only occasionally adopted, and always without general purposes, for instance by Navier in a note to Gauthey's *Traité de la construction des ponts* (Gauthey 1809, 1, 318–320) and by Lambel (1822) in a memoir on the stability of arches and retaining walls. On the contrary, in Mascheroni's *Nuove ricerche* the principle of virtual work assumes a central role and is used programmatically for the equilibrium analysis of rigid systems with one degree of freedom.

In this sense Mascheroni's contribution ideally belongs to the old tradition of the "science of weights" of Aristotelian origin, according to which, as pointed out by Sinopoli (2002, 2003), the equilibrium condition of the "simple machines" —real mechanisms subject to weights— was intended as a condition of non-activated motion by stating that the work of those weights must be zero for a (virtual) vertical displacement of their center of mass. Under this point of view, when an arch fails and transforms into a mechanism, it becomes a particular "machine" whose equilibrium condition can be derived by means of the principle of virtual work. As this "machine" is formed by a certain number of voussoirs that have absolute and relative (infinitesimal) movements, the great difficulty consists in correctly describing these movements in order to apply the principle.

Mascheroni's theoretical contribution mainly concerns the solution of this kinematical problem. He first gives a general discussion of the infinitesimal displacement of a segment, then demonstrates kinematical theorems that are finally applied to solve the problem of the *Equilibrio de' Rettilinei*, that is of systems with one degree of freedom formed by rigid bars connected with hinges and subject to weights. On this general basis the problem of the arch is easely solved since, at collapse, the vertical dislacements of the voussoirs coincide with those of the bars connecting the hinges about which the voussoirs move. If G and Q are the centers of mass of the voussoirs or of the bars (Fig. 7), from the principle of virtual work Mascheroni derives the equilibrium condition for the modes I and II (Mascheroni 1785, 25)

$$G\left(\frac{BT}{AF}-\frac{CE}{BE}\right)-Q\,\frac{CK}{BE}=0 \tag{14}$$

which can represent both $\delta L_{I}^{(a)}=0$ and $\delta L_{II}^{(a)}=0$ provided that we invert the position of the hinges at the intrados and extrados lines.

Monasterio (ca.1800)

For the theory of the arch the unpublished manuscript *Nueva teórica sobre el empuje de bóvedas* by the Spanish engineer Joaquin Monasterio is noteworthy for several reasons, as recently shown (Huerta and Foce 2003). Tackling the problem of the arch with a quite general approach, Monasterio first develops an original way of deriving the kinematically admissible mechanisms by observing that, when an arch fails, the voussoirs have movements of rotation and translation characteristic of the various collapse modes. Thus, by defining r and t as the rotations and translations of the voussoirs, Monasterio describes the collapse modes by means of proper sequences of the letters r and t, (proper in the sense that they correspond to admissible mechanisms). The number of letters gives the number of voussoirs in which the arch breaks at collapse; the order of letters, from left to right, indicates the types of movement to which the voussoirs are subject. Monasterio applies this procedure to non-symmetrical arches and finds seven collapse mechanisms with one degree of freedom. They are described by the sequences *tt*, *rrr*, *rrt*, *trr*, *tr*, *rt*, *trt* and shown if figures 1–7 of his plate 1, where the two sequences *tr* and *rt* are collapse modes with a composed movement of rotation and translation at a certain joint (Fig. 8).

Monasterio's approach to the kinematics of the arch is new and promising indeed. However, his results are not fully satisfactory, as can be shown by a different way of forming the sequences. Let us identify as R, T and RT the joints (not the

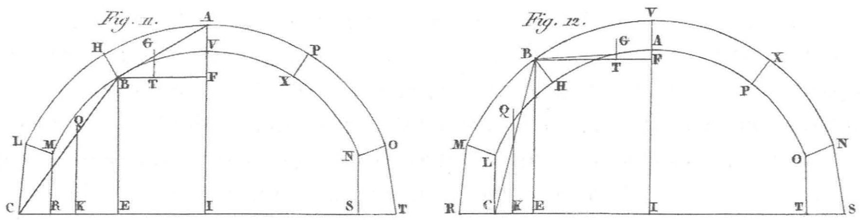

Figure 7
Sketches of the arches with the *rettilinei* for the analysis of Modes I and II (from Mascheroni 1785)

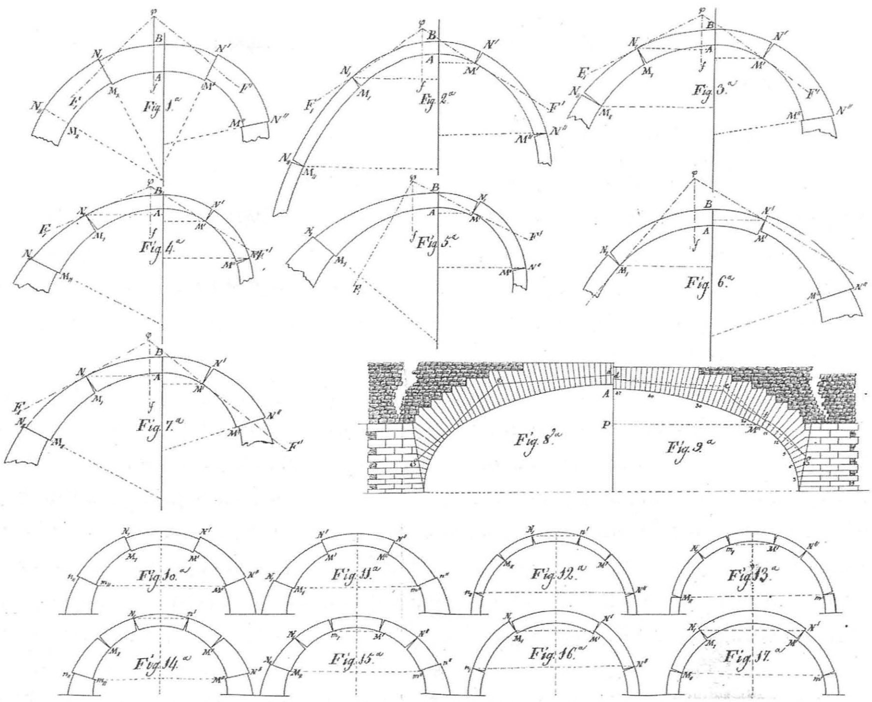

Figure 8
Monasterio's Plate 1 with the non-symmetric and symmetric collapse modes

voussoirs) where absolute and relative rotations, translations and roto-translations take place, so that the number of letters gives the number of rupture joints, and this number minus one gives the number of voussoirs at collapse. Through this choice it is easy to see that Monasterio's list is not complete because the rotation *r* or the translation *t* of a voussoir may derive from different types of absolute or relative movements *R*, *T* and *RT* at the rupture joints. As a matter of fact, there are twenty-three non-symmetrical collapse modes with one degree of freedom, represented by the following fourteen sequences, some of which stand for two opposite modes (Perazzo 2005):

T-T-T (corresponding to the sequence *tt*, figure 1 of Monasterio's plate 1)
T-R-R-T (corresponding to *trt*, figures 7 of plate 1)
T-T-R-R (two opposite modes)
R-R-R-T (two opposite modes) (valid fot both *rrt* and *trr* figure 3 and figure4 of plate 1)
R-R-R-R (corresponding to *rrr*, figure 2 of plate 1)
R-T-T-R
R-T-R-R (two opposite modes)
T-R-T-R (two opposite modes)
R-RT-R (two opposite modes)
RT-T-R (two opposite modes)
RT-R-R (two opposite modes)
RT-R-T (two opposite modes)
R-RT-T (two opposite modes) (valid for both *tr* and *rt*, figures 5 and 6 of plate 1)
RT-RT

For the symmetrical arch Monasterio does not write the sequences but provides the eight collapse modes drawn at the bottom of his plate 1. This number can rise to twelve if we consider that the modes of figures 12, 13, 14, 15, with two symmetrical hinges near the crown, may include four collapse modes with a single hinge at the extrados or intrados of the crown. Also in this case, however, Monasterio's list is not complete because there are twenty symmetrical modes with one degree of freedom, given by the following ten sequences representing two opposite modes (Perazzo 2005).

R-R-R-R-R (two opposite modes included in the modes of figures 12 and 13 of plate 1)
T-R-R-R-T (two opposite modes included in the modes of figures 14 and 15)
T-T-T-T (corresponding to figures 10 and 11)
R-T-R-T-R

R-RT-RT-R (corresponding to figures 16 and 17)
RT-R-R-RT
RT-R-RT
R-R-R-R-R-R (corresponding to figures 12 and 13)
T-R-R-R-R-T (corresponding to figures 14 and 15)
R-T-R-R-T-R

A second relevant feature of Monasterio's memoir comes from the fact that he tackles the stability analysis starting with the non-symmetrical arch and then adapting it to the special case of the symmetrical arch. Further, the way of deriving the collapse stability is proof of the originality of his analysis. For instance, the condition for the activation of the non-symmetrical mode *rrr* (*R-R-R-R* with our notation) is obtained by imposing necessary static requirements, that is (Fig. 9):

1) the right component of the weight must go through the intrados edge *C*;
2) the left component of the weight must go through the extrados edge *B*;
3) the moment of the left component of , with respect to the intrados edge *A* must be lower than the moment of the weight with respect to the same point;
4) the moment of the right component of with respect to the extrados edge *D* must be greater than the moment of the weight with respect to the same point.

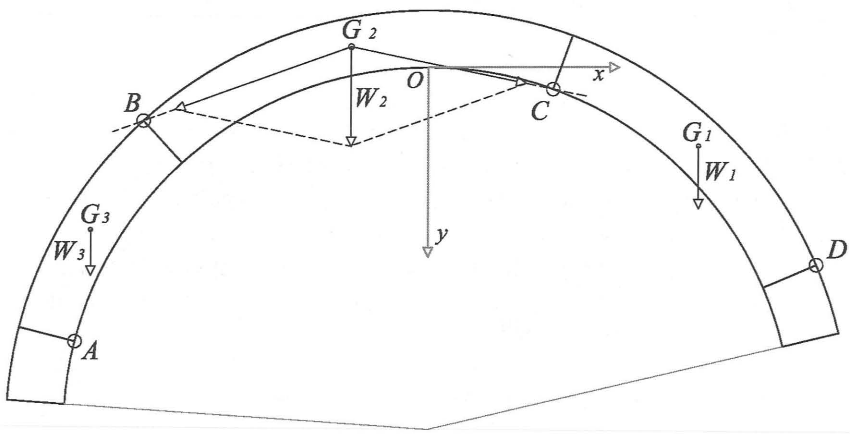

Figure 9
Monasterio's analysis of the rotational collapse mode of a non-symmetric arch (redrawn from Monasterio)

By treating these conditions Monasterio arrives at a stability disequality which he adapts to the symmetric arch. Thus, by taking the origin of the axes at the crown intrados, he correctly finds that collapse Mode I of figure 10 a cannot occur if (in our notations)

$$(W_1 + W_2)\frac{x_E - x_G}{y_E + s} - W_2\frac{x_D - x_{G_2}}{y_D + s} \geq 0 \tag{15}$$

and, similarly, that collapse Mode II of figure10b cannot occur if

$$W_2\frac{x_D - x_{G_2}}{y_D} - (W_1 + W_2)\frac{x_E - x_G}{y_E} \geq 0 \tag{16}$$

Now, the two terms in (15) and in (16) are nothing other than the thrusts $H^{r,e}_{\max}$ and $H^{r,e}_{\min}$ and the thrusts $H^{r,i}_{\max}$ and $H^{r,i}_{\min}$, respectively. Thus by searching for the minimum of the first terms of (15) and (16) and the maximum of the latter ones we obtain two stability disequalities which become (7) and (9) in the case of limit equilibrium.

Monasterio uses (15) for the semicircular arch of constant thickness under its own weight. By trial and error he finds that the minimum thickness is between 1/8 = 0.125 and 1/8 = 0.111 of the intrados radius and the rupture joint at the haunches is between 54° and 56° from the crown. This result is quantitatively correct and agrees with the calculation by Petit (1835), who gave 0.114. Mi-

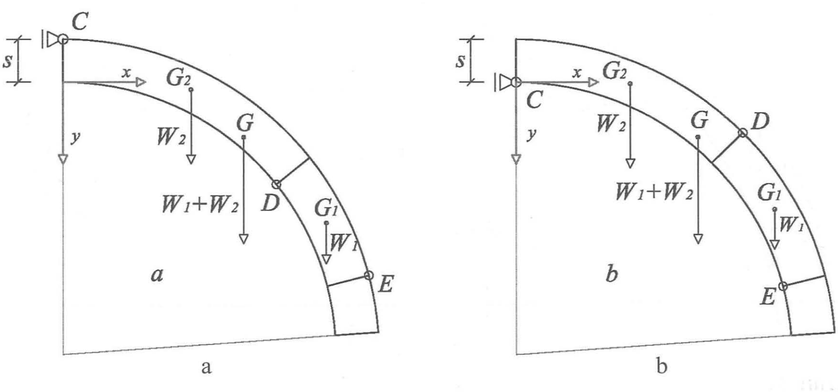

Figure 10
Monasterio's analysis of stability of a symmetric arch with respect to Modes I and II

lankovitch (1907) obtained the rigorous value 0.1136 corresponding to the rupture joint at 54°29’ from the crown.

Persy (1825), Navier (1826), Michon (1857)

The contributions by Persy, Navier and Michon on the collapse of the arch are directly connected with Coulomb’s method of maxima and minima and represent an important theoretical improvement of Coulomb’s results. Other authors, both before and after them, have taken Coulomb’s method as a point of departure, even though with less general purposes. In this sense we can cite the works of Berard (1810), Audoy (1820), Lamé and Clapeyron (1823), and the “special issue” of the *Mémorial de l’Officier du Génie* of 1835, where three long memoirs are devoted to particular applications of the method (Garidel and Petit) and to a graphic calculation of the extreme values of the thrust (Poncelet).

Persy’s treatment of the matter is particularly enlightening because he intentionally starts from Coulomb’s analysis to point out its deficiency regarding the point of application of the thrust at the crown joint. He initially considers the thrust applied at a generic point of the crown and finds the two extreme values max $H^r_{\min}$ and min $H^r_{\max}$, to which he associates the joints J and j, respectively. To conceive a kinematically admissible mechanism when the necessary condition of collapse max $H^r_{\min}$ = min $H^r_{\max}$ is attained, he admits for a moment that the thickness at the crown shrinks to a single point and concludes that if the joint J is higher than j, the collapse mode of figure11a may occur and, conversely, if the joint J is lower than j, the collapse mode of figure 11 (b) may occur.

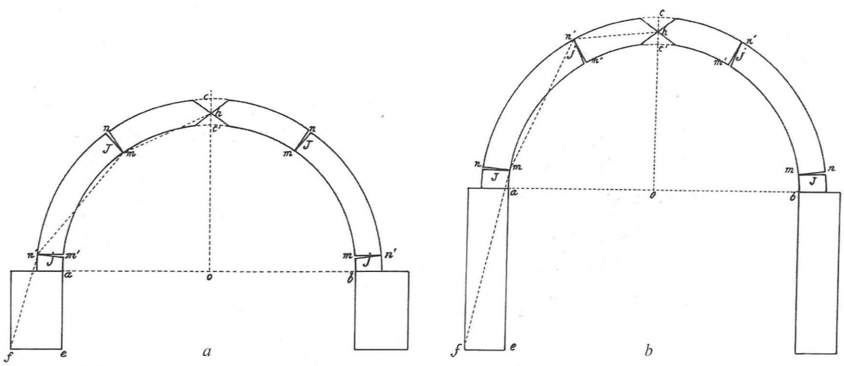

Figure 11
Persy’s two rotational collapse modes in the hypothetical case of a point contact at the crown joint

Bearing in mind the two opposite rotational modes shown in figure 3 and figure 4 with a hinge at the crown, Persy observes that when the crown joint has a finite thickness, the point of application of the thrust may assume two limit positions corresponding to the crown extrados and the crown intrados. Thus, to obtain the collapse mode I of figure 3, with thrust at the crown extrados, he defines the new extreme values max $H^{r,e}_{\min}$ and min $H^{r,e}_{\max}$ and concludes that mode I may occur when max $H^{r,e}_{\min}$ = min $H^{r,e}_{\max}$ under the kinematical condition that the joint corresponding to max $H^{r,e}_{\min}$ is higher than the joint corresponding to $H^{r,e}_{\max}$; conversely, to obtain the collapse mode II of figure 4, with thrust at the crown intrados, he defines the new extreme values max $H^{r,i}_{\min}$ and min $H^{r,i}_{\max}$ and concludes that the mode II may occur when max $H^{r,i}_{\min}$ = min $H^{r,i}_{\max}$ under the condition that the joint corresponding to max $H^{r,i}_{\min}$ is lower than the joint corresponding to min $H^{r,i}_{\max}$.

Persy's analysis is correct in so far as collapse modes I and II shown in figure 3 and figure 4 are concerned. In this sense, Persy clarifies Coulomb's discussion since he introduces two pairs of extremes values of the thrust corresponding to the application point at the crown extrados and intrados and gives the collapse conditions (7) and (8) for mode I and (9) and (10) for mode II. These results were obtained independently by Navier (1826) and received a clear exposition in a later work by Michon (1857). Nevertheless they do not completely solve the collapse analysis of the symmetrical arch as they cannot include the general form of the rotational modes shown in figure1 and figure 2.

Durand-Claye (1867, 1868, 1880)

The progress of the studies on the elasticity and strength of materials during the first decades of the ninteteenth century had an indirect influence on the theory of the arch as well. Starting from the 1839s, the traditional approach in terms of collapse analysis soon appeared out of date in the face of new questions concerning the actual stresses within the arch and, for at least forty years, the problem of finding the "true" line of thrust seriously troubled the minds of generations of scholars before the elastic approach was recognized and accepted as the only rational way out. During these forty years many methods were proposed in order to remove the statical indetermination of the problem on the basis of arbitrarious assumptions regarding the point of application of the thrust at the crown or by means of metaphysical principles concerning the features of the actual line of thrust. The tragicomical result was what the Italian elastician Francesco Crotti denounced as a dizziness of the minds in a noteworthy *Esame critico* (Crotti 1875) written in reaction to the *n*th attempt at solving the problem by means of *a priori* hypotheses.

In this contradictory phase of the theory of the arch an important contribution was made in 1867 by Alfred Durand-Claye with the method of the areas of stability (Foce and Aita 2003). Durand-Claye was perfectly aware that the true thrust line of a stable arch is statically indeterminate. Thus, instead of searching for the "actual" thrust line, he elaborates a general method for determining all the admissible thrust lines which fulfill the equilibrium equations (with respect also to the strength of materials): in the case of a stable arch the true one will necessarily be included. As Durand-Claye himself writes, "de la possibilité de l'équilibre, nous concluons à la stabilité" (Durand-Claye 1867, 65).

Briefly, and taking into account only the part of the method dealing with the rotational equilibrium, Durand-Claye considers a voussoir of a symmetric arch and writes the equations of the thrusts $H^r_{\min}$ and $H^r_{\max}$, that is

$$H^r_{\min}(\phi, y) = \frac{W(\phi)[x_M(\phi) - x_G(\phi)]}{y - y_M(\phi)} \tag{17}$$

and

$$H^r_{\max}(\phi, y) = \frac{W(\phi)[x_N(\phi) - x_G(\phi)]}{y - y_N(\phi)} \tag{18}$$

where the vertical distance y is the independent variable defining the point of application of the thrust at the vertical joint and the differences of the coordinates are the lever arms of weight and thrust with respect to the intrados M and the extrados N of the joint at angle ϕ (Fig. 12 a). In the plane Hy these equations represent two equilateral hyperbolas with a common vertical asymptote coinciding with the line of the crown joint and with horizontal asymptotes given by the straight lines through the intrados M and the extrados N of the joint, respectively. Given ϕ, the admissible values of the crown thrust are graphically represented by the area bounded by the two hyperbolas and the two horizontal straight lines starting from the extrados and intrados edge of the crown joint. Now, by drawing the hyperbolas (17) and (18) for each joint of the arch and taking the common area to all the areas previously defined, Durand-Claye obtains the so-called *area of stability*, which in the plane Hy is the locus of the points representing the admissible values of the thrust and their point of application at the crown for the rotational equilibrium of the whole arch (Fig. 12 b).

If, for a certain value of the thickness, the area of stability shrinks to a point, then the collapse condition is attained since a unique admissible thrust exists and, at the same time, a certain collapse mode becomes kinematically

admissible, with hinges located at the intrados or extrados of the joints corresponding to the intersecting hyperbolas. This case is shown in figure 13 (a) and corresponds to the collapse Mode I of a semicircular arch subject to its own weight and bearing a horizontal fill with the same specific weight as the arch,

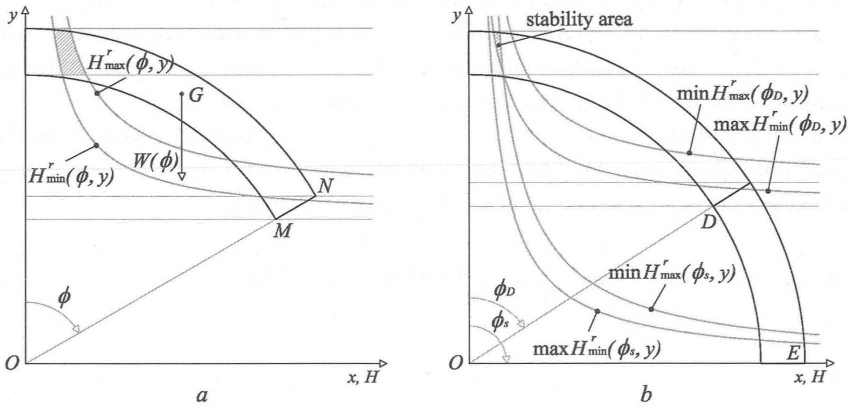

Figure 12
Area of the admissible thrusts for the rotational equilibrium of a generic voussoir (a) and area of stability for the whole arch (b)

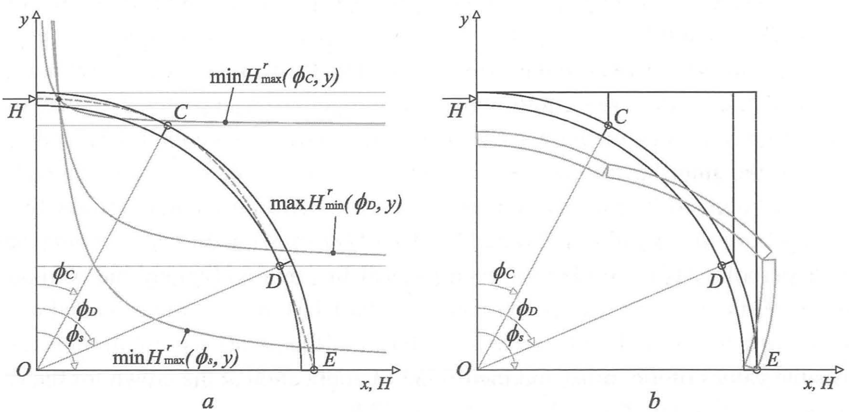

Figure 13
Collapse condition according to Durand-Claye's method (a) and corresponding collapse mode (b)

with thickness $s = 0.0481r$ and hinges at $\phi_C \cong 28°$, $\phi_D \cong 67°$ and at the springing (Fig. 13 b).

Scheffler (1857), Ceradini (1873, 1887)

Our interest in Scheffler's work concerns his study of the geometrical properties of the lines of maximum and minimum thrust, probably the first general analysis considering both symmetrical and non-symmetrical arches. Scheffler clearly recognizes that, for a stable arch, the statically admissible thrust lines are bounded by two limit lines corresponding to the minimum and maximum value of the horizontal thrust H and that the limit condition of equilibrium is attained when a line of thrust can be draw within the arch ring which has, at once, the property of the minimum and maximum thrust (Scheffler 1857, 48).

As far as the line of minimum thrust, Scheffler considers the thrust line through the extrados of the crown joint and the intrados of the springing and asserts that:

1) If this line lies within the arch ring, then it is the line of minimum thrust (Fig. 14 a);
2) If this line cuts the intrados, but not the extrados, then the line of minimum thrust goes through the extrados of the crown and touches the intrados at a certain point near the springing (Fig. 14 b);
3) If this line cuts the extrados, but not the intrados, then the line of minimum thrust touches the extrados at a certain point (starting from a point internal to the crown joint) and goes through the intrados of the springing (Fig. 14 c);
4) If this line cuts both the extrados and intrados, then the line of minimum thrust touches the intrados at a certain point (starting from a point internal to the crown joint) and touches the intrados at a certain point near the springing (Fig. 14 d);

Similarly, for the line of maximum thrust Scheffler considers the thrust line through the intrados of the crown joint and the extrados of the springing and asserts that:

1) If this line lies within the arch ring, then it is the line of maximum thrust (Fig. 15 a);
2) If this line cuts the intrados, but not the extrados, then the line of maximum thrust touches the intrados at a certain point (starting from a point internal to the crown joint) and goes through the extrados of the springing (Fig. 15 b);

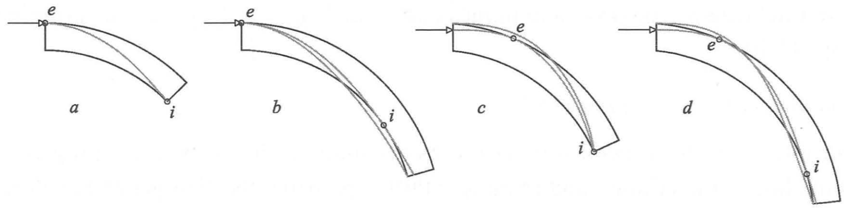

Figure 14
Possible positions of the line of minimum thrust (redrawn from Scheffler 1857)

3) If this line cuts the extrados, but not the intrados, then the line of maximum thrust goes through the intrados of the crown and touches the extrados at a certain point over the springing (Fig. 15 c);
4) If this line cuts both the extrados and intrados, then the line of maximum thrust touches the intrados at a certain point (starting from a point internal to the crown joint) and the extrados at a certain point near the springing (Fig. 15 d);

The previous discussion was given also by Ceradini (1873), who extended Scheffler's analysis on the basis of the following general statement: if two lines of thrust intersect, their points of intersection lie on the straight line connecting the point of intersection E of the two reactive systems at the left springing with the point of intersection F of the two reactive systems at the right springing (Fig. 16).

This statement is a direct consequence of the construction of two funicular polygons for the same external load. On its basis Ceradini shows all the possible relative positions of the thrust lines corresponding to different reactive systems and demonstrates the properties of the lines of maximum and minimum thrust for

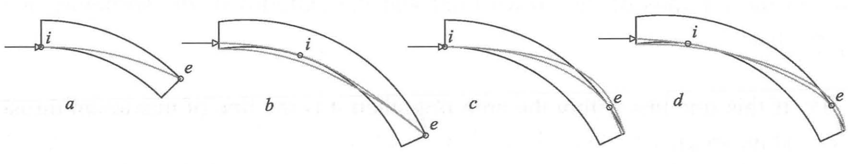

Figure 15
Possible positions of the line of maximum thrust (redrawn from Scheffler 1857)

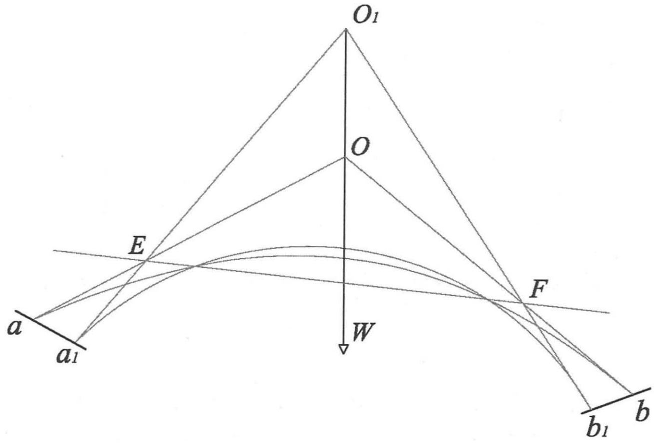

Figure 16
Ceradini's statement on the lines of thrust and the corresponding reactive systems (from Ceradini 1873)

a non-symmetrical arch under a vertical load. In particular, the line of minimum thrust necessarily touches the extrados at one point *e* and the extrados at two points *i* as in figure 17 (a), while the line of maximum thrust necessarily touches the intrados at one point *i* and the extrados at two points *e* as in figure 17 (b).

If a line of thrust has two points of contact with the extrados and two points of contact with the intrados located as in figure 18, it has the features of both the lines of maximum and minimum thrust and then it is the only statically admissible line.

Moreover, Ceradini shows that the same properties hold if the resultant of the external load is not vertical, with the difference that they refer to the component of the reactive systems which acts perpendicularly to that resultant (Ceradini 1887).

Conclusions

In Swain's treatise of 1927 no mention is made of the authors quoted above and is probable that their contributions were rather extraneous to his scientific training. In spite of that, it is not difficult to recognize a deep convergence between his approach and the results of the pre-elastic studies that we have discussed in the previous section. As a matter of fact, Swain's peroration in favour of a "weighted"

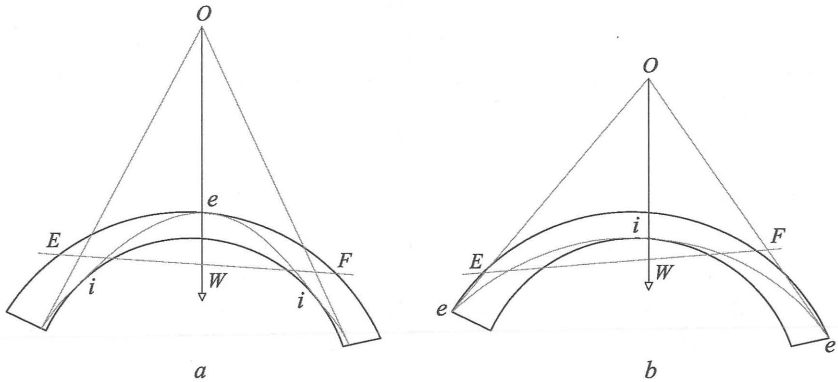

Figure 17
Lines of minimum and maximum thrust for a non-symmetrical arch (from Ceradini 1873)

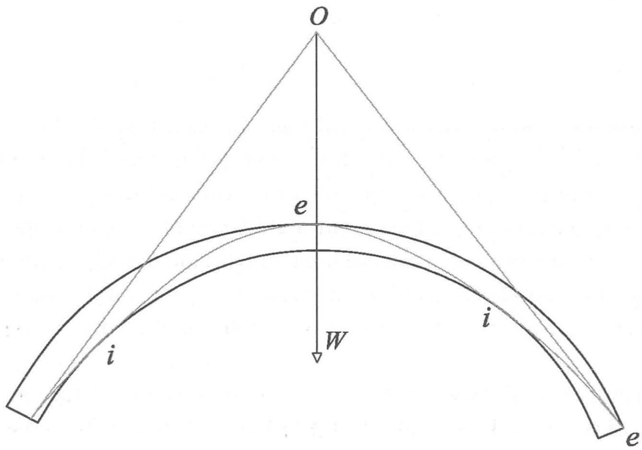

Figure 18
Line of minimum and maximum thrust for a non-symmetrical arch (from Ceradini 1873)

use of the elastic methods for the analysis of the masonry arch has the value of a methodological choice whose last consequences lead to the structural philosophy of limit analysis that, after Heyman's lesson, is nowadays considered as the basis for the study of the stone skeleton. In this sense, also the lesson from history should be carefully attended.

Acknowledgments

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Nondeterministic structural analyses: Insights on static behavior of hyperstatic structures

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The 1826 publication of Navier's *Leçons*, the design of early suspension bridges, and the design of iron box girder railroad bridges exemplify the transforming changes in the design of structures that began in the first half of the nineteenth century. Navier's text demonstrates that linear elastic structural analyses provide essential insights on structural behavior and may serve as bases for an allowable stress design approach. For example, Navier gives the general solution for the forces in a king-post truss and observes that a parallel-chord truss may be viewed as a beam with an effective moment of inertia proportional to the chord areas times the square of the distance between the chords. This basic analogy formed the basis for the design of truss chords in the US in the 1830's. Navier also provides an analytical solution to what may be viewed as the "elementary hyperstatic system", the three bar truss shown as figure 112 in plate IV of his text. Navier writes the joint equilibrium equations, the linear elastic force-deformation equations, and the small-displacement geometric compatibility equations between the element axial deformations and the joint displacements. Navier then uses a "stiffness formulation" to solve for the joint displacements and the member forces. Navier in all likelihood understood the increase in computational effort required for analyses of trusses with many joints. Implicit in Navier's analysis are deterministic (zero) displacement boundary conditions and the assumption that the members fit perfectly. That is, Navier did not consider any self-equilibrated forces in the hyperstatic truss prior to the application of the load.

The early suspension bridges built in Britain, France and the United States posed challenging new design problems for engineers. Small-span suspension

bridges have live-load-to-dead-load ratios significantly larger than, say, those of stone arches. Moreover, hanging cables or chains have small vertical stiffnesses for some live load conditions. Therefore design live loads models were debated intensely. Navier argued for a uniform design live load of three 65 kg persons per square meter, or 200 kg/m^2, while Marc Seguin reasoned that in cases where the population was small and the traffic light, "it is quite possible to lower the requirement, as the probability of such a load is so small as to be considered non-existent" (Peters 1987). Seguin's concept of reducing the live load based on the probability of occurrence continues today in both building and bridge design. Designers agreed that adequate stiffness was a critical design criterion and numerous conceptual designs of stiffening systems were proposed and tried (Gasparini et al. 1999). Navier's advanced understanding of the effect of the cable axial force on its vertical stiffness formed his judgment that stiffness is most easily achieved by using small sag-to-span ratios. In addition to issues of appropriate loads and stiffness, suspension bridge design also initiated discussions on reliability, specifically on the relative short and long term reliabilities of chains versus wire cables. Navier did not favor wire cables because he believed that they were susceptible to corrosion over time. On the other hand, designers understood that the strength of a chain depends on its weakest link.

Robert Stephenson's concept of riveted iron box girder railroad bridges raised other design issues. For one, the stability of plates in compression had to be assured. Most importantly, there arose a need for a rational basis for design of riveted connections. A riveted (or bolted) connection is the *bete noir* of a linear elastic analysis/design approach. Stress states depend on unknown boundary conditions and on friction from the prestressing of the rivets and plates. Fairbairn (1849) and Hodgkinson advocated a strength design approach; that is, proportioning a connection such that its strength is safely greater than working loads (or, say, equal to factored loads). Stephenson and Clark (1850) advocated designs that gave satisfactory performance, say no slip, at service loads. Since it was not possible to predict strength analytically, Fairbairn and Hodgkinson performed extensive tests, which revealed considerable statistical scatter in strengths. Fairbairn based his "100–75–56" rule on observed *mean* strengths.

These and other nineteenth century design experiences with new materials and structural forms made engineers more aware of uncertainties in loads and strengths and of the need to provide both sufficient strength and satisfactory performance at service conditions. In the absence of analytical methods and testing resources for predicting strength, design methods based on linear elastic analyses became dominant. These, of course, involve defining very conservative "working" loads, performing linear elastic analyses, and "allowing" stresses that are

fractions of very conservative ("minimum") strengths. Such methods have produced safe designs and remain predominant. The computational effort of performing analyses was decreased by manual iterative methods for solving simultaneous equations and then revolutionized by computer-based finite element formulations. However, Heyman (1995) notes that computed stresses in some hyperstatic systems are "illusory", because of uncertain force and displacement boundary conditions. Further, Heyman observes that such uncertainties generally do not affect the strength of systems and therefore he advocates the use of mechanism limit state analyses for certain systems.

Within the past forty or so years, considerable research has been performed on nondeterministic analysis and design methods. Such methods provide a rational basis for defining design criteria for exceptional projects such as "lifelines", power plants, etc. Because some of the uncertainty is modeled, nondeterministic methods have the potential to improve decision-making and design. Estimating reliability and achieving a design with a prescribed reliability are rational design objectives. The objective here is not to provide a chronological account of individual developments in nondeterministic methods. Rather, it is to present basic nondeterministic methods for static structural analysis, discuss insights that they may provide, contrast them with deterministic methods, and discuss possible associated design methods. Navier's elementary hyperstatic system, with specific direction angles as shown in figure 1, is used to illustrate ideas and concepts. Deterministic linear elastic analyses are reviewed first. Then three nondeterministic linear elastic analysis approaches are discussed. Next, deterministic and nondeterministic mechanism limit state analyses are compared, considering both ductile and brittle elements.

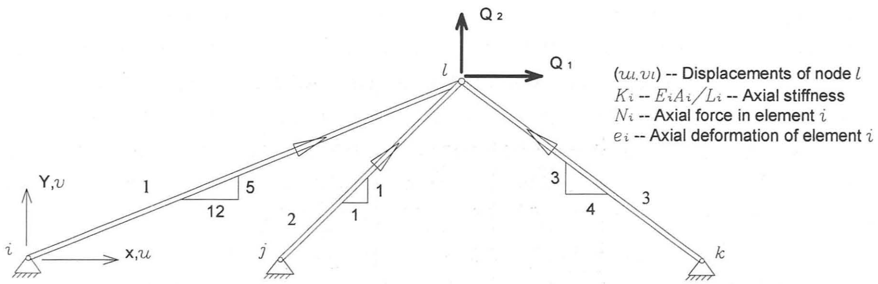


Figure 1
Navier's elementary hyperstatic system with specific direction angles

Linear elastic analyses. Deterministic systems and loads

The most common method for linear elastic static analysis involves writing nodal equilibrium equations in the matrix stiffness form:

$$\mathbf{KU} = \mathbf{P} \tag{1}$$

In which **K** is the system stiffness matrix, **U** is the vector of nodal displacements, and **P** is the vector of effective nodal loads. If the system is considered to be deterministic, **K** is a matrix of real numbers. **K** is typically assembled by adding stiffness contributions from individual elements; this process reveals that the stiffness of elements or subsystems in parallel are additive. The effective nodal load vector, **P**, may be expressed as:

$$\mathbf{P} = \mathbf{MQ} \tag{2}$$

in which **Q** is a vector of magnitudes for any specific load system and **M** is a matrix whose rows are contributions to effective nodal loads from unit values of the forces in **Q**. Figure 2 shows example **Q** vectors for a truss and a plane frame.

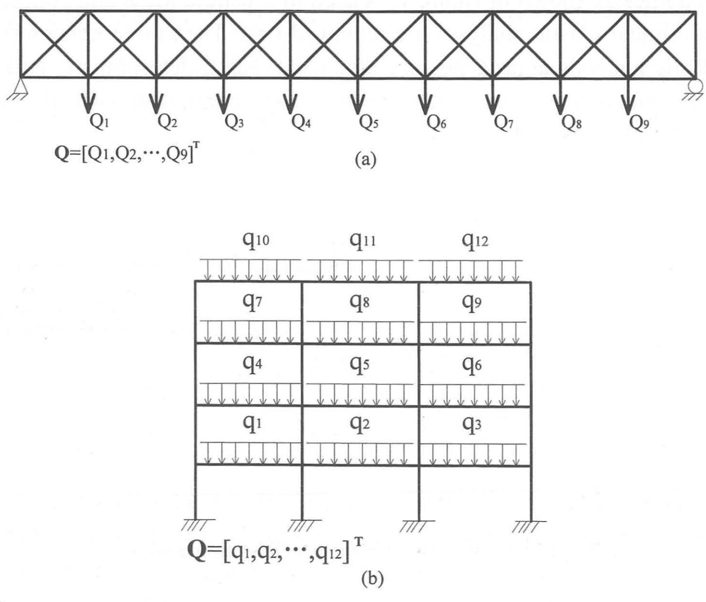


Figure 2
Load magnitude vectors, **Q**, for two specific load systems

After imposing displacement boundary conditions, the nodal displacement vector is, symbolically:

$$\mathbf{U} = (\mathbf{K}^{-1}\mathbf{M})\mathbf{Q} \tag{3}$$

In turn, any vector of element stress resultants, **N**, may be expressed as a linear function of **U**,

$$\mathbf{N} = \mathbf{B}\mathbf{U} = (\mathbf{B}\mathbf{K}^{-1}\mathbf{M})\mathbf{Q} = \mathbf{A}_q\mathbf{Q} \tag{4}$$

Eqs. (3) and (4) simply indicate that for a linear elastic system, any load effect vector may always be expressed as a linear function of a load magnitude vector, **Q**. The matrices $\mathbf{K}^{-1}\mathbf{M}$ and $\mathbf{B}\mathbf{K}^{-1}\mathbf{M} = \mathbf{A}_q$ may have important physical meaning for specific load systems. For example, for the **Q** vector shown in figure 2a, $\mathbf{K}^{-1}\mathbf{M}$ is a matrix whose rows are influence line values for nodal displacements and $\mathbf{B}\mathbf{K}^{-1}\mathbf{M} = \mathbf{A}_q$ is a matrix whose rows are influence line values for element axial forces. For the **Q** vector of figure 2b, the rows of $\mathbf{B}\mathbf{K}^{-1}\mathbf{M} = \mathbf{A}_q$ show the contributions of each bay loading to the stress resultants in **N**. Therefore the signs of the terms in any row of $\mathbf{A}_q$ indicate which bays must be loaded to obtain extreme values of the corresponding stress resultant. For the Navier system shown in figure 1, the matrix **M** is simply a unit diagonal matrix and a simple stiffness analysis gives:

$$\mathbf{U} = \begin{Bmatrix} u_l \\ v_l \end{Bmatrix} = (\mathbf{K}^{-1}\mathbf{M})\mathbf{Q} = \frac{1}{1.8672\mathrm{K}} \begin{bmatrix} 1.0079 & -0.3750 \\ -0.3750 & 1.9921 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \mathrm{Q}_1 \\ \mathrm{Q}_2 \end{Bmatrix} \tag{5a}$$

$$\mathbf{N} = \begin{Bmatrix} \mathrm{N}_1 \\ \mathrm{N}_2 \\ \mathrm{N}_3 \end{Bmatrix} = \mathbf{B}\mathbf{U} = \mathrm{K} \begin{bmatrix} 12/13 & 5/13 \\ 1/\sqrt{2} & 1/\sqrt{2} \\ -4/5 & 3/5 \end{bmatrix} \begin{Bmatrix} u_l \\ v_l \end{Bmatrix} \tag{5b}$$

$$\mathbf{N} = \begin{bmatrix} 0.4211 & 0.2249 \\ 0.2397 & 0.6124 \\ -0.5524 & 0.8008 \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \mathbf{A}_q\mathbf{Q} \tag{5c}$$

In which K = EA/L is the axial stiffness of all three elements. If a structure is statically determinate, $\mathbf{A}_q$ is independent of material and element properties. Although the truss is hyperstatic, the element axial forces are independent of the element axial stiffnesses because all the axial stiffnesses are the same. Of course if the element axial stiffnesses were not equal, any element axial force (in a

hyperstatic system) is proportional to that element's stiffness contribution. The linear elastic analysis represented by Eqs. (5) assumes zero lack-of-fit and zero prescribed initial support displacements. But both lack-of-fit and non-zero prescribed support displacements may be considered as actions or loads. Let $\mathbf{e_0} = [e_{10}\ e_{20}\ e_{30}]^T$ be a vector of initial element axial deformations and $\mathbf{U_0} = [u_{i0}\ v_{i0}\ u_{j0}\ v_{j0}\ u_{l0}\ v_{l0}]^T$ be a vector of non-zero prescribed support displacements. Then a linear elastic analysis gives:

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} = K \begin{bmatrix} -0.5248 & 0.4568 & -0.2017 \\ 0.4568 & -0.3975 & 0.1757 \\ -0.2019 & 0.1757 & -0.0776 \end{bmatrix} \mathrm{e}_0 +$$

$$+K \begin{bmatrix} -0.4844 & -0.2018 & 0.3230 & 0.3230 & 0.1615 & -0.1201 \\ 0.4217 & 0.1757 & -0.2811 & -0.2811 & -0.1406 & 0.1054 \\ -0.1863 & -0.0776 & 0.1242 & 0.1242 & 0.0616 & 0.0462 \end{bmatrix} \mathbf{U}_0 \qquad (6a)$$

$$\mathbf{N} = \mathbf{A_e}\,\mathbf{e_0} + \mathbf{A_u}\,\mathbf{U_0} \qquad (6b)$$

The matrices $\mathbf{A_e}$ and $\mathbf{A_u}$ must be identically zero for statically determinate systems. For the hyperstatic Navier truss, the axial forces from $\mathbf{e_0}$ and $\mathbf{U_0}$ are proportional to the element axial stiffnesses, $K_i = K$. For a linear elastic system, the effects of the three actions may be superposed:

$$\mathbf{N} = \mathbf{A_e}\,\mathbf{e_0} + \mathbf{A_u}\,\mathbf{U_0} + \mathbf{A_q}\mathbf{Q} \qquad (7)$$

Effects of changes in temperature may also be added to Eq. (7). It should be noted that, thus far, no models for the action vectors, $\mathbf{Q}$, $\mathbf{e_0}$, and $\mathbf{U_0}$ have been introduced. That is, Eqs. (1) to (7) are valid for deterministic system models and for *both* deterministic and nondeterministic load or action models. For deterministic linear elastic analyses, the actions are modeled as vectors of real numbers. Then any response vector will also be a vector of real numbers. For example, if $\mathbf{Q} = [10\ 50]^T$ and K= 30,000, then $\mathbf{U} = [0.000155\ 0.00171]^T$ and $\mathbf{N} = [15.46\ 33.01\ 34.53]^T$.

Observations. Deterministic linear elastic analyses give "point estimates" of responses of *models* of structures to *models* of loads or actions. Whether the estimates truly predict actual structural response depends on the quality or "goodness" of the system and load models. A designer may always try to refine or improve the structural system model. For example, if there are concerns about the rotational stiffness of the nodes or, say, the flexibility of the supports, a frame model may be defined as shown in figure 3.

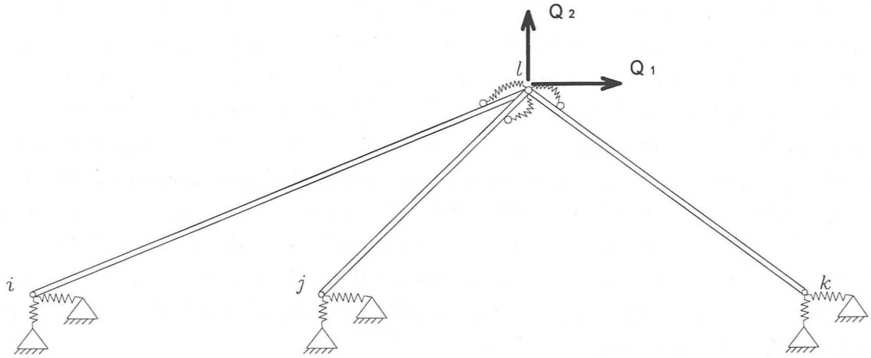


Figure 3
Navier system model with support flexibilities and rotational stiffness of connection at node l

Deterministic linear elastic analyses with refined or improved models still give only point estimates of responses. Deterministic linear elastic analyses do provide checks of system and element serviceability criteria and indicate rational ways of increasing system stiffness or decreasing system responses. Deterministic linear elastic analyses do not provide information on strength, except, if equilibrium is satisfied with stresses at or below yield and no mechanism exists, actual strengths of ductile systems must be greater than the working loads used for linear elastic analysis. Deterministic linear elastic analyses do not provide information on the possible range of responses. If the linear elastic material model is appropriate, linear elastic analyses may be applied to most systems using available finite element programs. However, there are systems such as bolted connections, welded connections, cracked masonry, or cracked concrete systems where uncertainties in the details or boundary conditions make linear elastic analyses "nonpredictive."

Linear elastic analyses. Deterministic systems, nondeterministic loads

As noted, Eqs. (1) to (7) are valid for linear elastic deterministic systems and for both deterministic and nondeterministic load models. Three principal nondeterministic models for a scalar load, Q_i, are shown in figure 4. Figure 4b depicts a random variable model; figure 4c depicts an interval number (Alefeld 1983) model; and figure 4d depicts a fuzzy number (Zadeh 1965; Kaufmann and Gupta

1985) model. To define vector random variables, multidimensional probability density functions are generally required. Multidimensional probability density functions are difficult to define and use; however, there are two important exceptions. If a set of random variables are independent, then a multidimensional probability density function is simply a product of one-dimensional density functions. If the random variables are normal or Gaussian, then the multidimensional probability density function is fully defined by two statistical moments, the mean and covariance matrices. Even if the random variables are not independent or Gaussian, it is common to perform analyses with only these two moments, using "second moment" algebra. The mean and covariance matrices of a random vector, **Q**, are defined by:

$$\text{Mean matrix} = E[\mathbf{Q}] = \boldsymbol{\mu}_{\mathbf{Q}} \tag{8}$$

$$\text{Covariance matrix} = E[(\mathbf{Q} - \boldsymbol{\mu}_{\mathbf{Q}})(\mathbf{Q} - \boldsymbol{\mu}_{\mathbf{Q}})^T] = \boldsymbol{\Sigma}_{\mathbf{QQ}} \tag{9}$$

In which E[ ] denotes the expectation operator.

If only the above two statistical moments are known, **Q** is referred to as a "second moment" vector, and is denoted as: $\mathbf{Q} \sim (\boldsymbol{\mu}_{\mathbf{Q}}; \boldsymbol{\Sigma}_{\mathbf{QQ}})$. A generic term of the covariance matrix is: $\text{Cov}\,[Q_iQ_j] = E\,[(Q_i - \mu_{Qi})\,(Q_j - \mu_{Qj})]$. The $\text{Cov}\,[Q_iQ_j]$ may

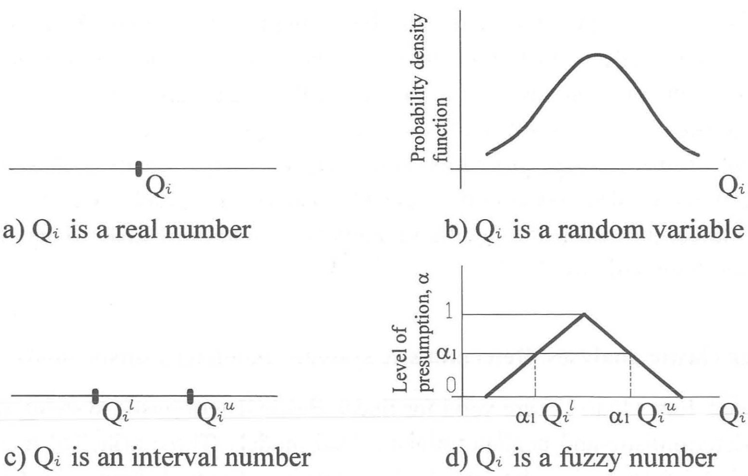


Figure 4
The deterministic and three nondeterministic models for a scalar load, Q_i

be expressed in terms of a correlation coefficient, ρ_{QiQj}, between Q_i and Q_j as follows:

$$\mathrm{Cov}\,[Q_iQ_i] = \rho_{Q_iQ_j} \sqrt{\mathrm{Var}[Q_i]}\,\sqrt{\mathrm{Var}[Q_j]} \tag{10}$$

$\rho_{Q_iQ_j}$ is a measure of the linear correlation between two random variables, a concept not contained in deterministic analyses. It follows from Eq. 5c that a linear elastic analysis of a deterministic system is simply a linear transformation between a load vector, **Q**, and a response vector, **N**. From properties of the expectation operator, the mean and covariance matrices of the response vector are:

$$E[\mathbf{N}] = \boldsymbol{\mu}_N = \mathbf{A_q}\,\boldsymbol{\mu}_\mathbf{Q} \tag{11}$$

$$E[(\mathbf{N} - \boldsymbol{\mu}_N)(\mathbf{N} - \boldsymbol{\mu}_N)^T] = \boldsymbol{\Sigma}_{NN} = \mathbf{A_q}\,\boldsymbol{\Sigma}_\mathbf{QQ}\,\mathbf{A_q}^\mathbf{T} \tag{12}$$

Therefore the mean and covariance matrices of any response vector of a linear elastic deterministic system are found using simple matrix multiplication. For example, assume the load vector for the Navier system of figure 1 is defined by:

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \mathbf{Q} \sim \left(\begin{Bmatrix} 10 \\ 50 \end{Bmatrix}; \begin{bmatrix} 4 & 15 \\ 15 & 225 \end{bmatrix} \right) \tag{13}$$

Then, using Eqs.11 and 12, and assuming K = 30.000., **U** and **N** are second moment vectors:

$$\begin{Bmatrix} u_l \\ v_l \end{Bmatrix} = \mathbf{U} \sim \left(\begin{Bmatrix} 0.000155 \\ 0.00171 \end{Bmatrix}; \begin{bmatrix} 0.7767 \times 10^{-8} & -4.378 \times 10^{-8} \\ -4.378 \times 10^{-8} & 27.76 \times 10^{-8} \end{bmatrix} \right) \tag{14}$$

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} = \mathbf{N} \sim \left(\begin{Bmatrix} 15.46 \\ 33.01 \\ 34.53 \end{Bmatrix}; \begin{bmatrix} 14.94 & 36.08 & 42.81 \\ 36.08 & 89.00 & 107.67 \\ 42.81 & 107.67 & 132.31 \end{bmatrix} \right) \tag{15}$$

The coefficient of variation of a random variable is defined as vVar$[N_i]$/ $E[N_i]$. The coefficients of variation of $\mathbf{N} = [N_1\ N_2\ N_3]^T$ are 0.25, 0.285, and 0.333 respectively. In general, each response has a different uncertainty. The moments of **N** given in Eq. 15 were computed for the assumed correlation between Q_1 and Q_2 contained in the covariance matrix in Eq. 13. $\boldsymbol{\Sigma}_{NN}$ of Eq. 15 indicates that the

three axial forces are highly correlated. The effects of changing the correlation between Q_1 and Q_2 on the moments of **N** are easily quantified using Eq. 12. Even if the loads are uncorrelated, the axial forces are correlated because they are all functions of the same loads. In other words, a diagonal load covariance matrix, $\mathbf{\Sigma}_{\mathbf{QQ}}$, still results in a full response covariance matrix, $\mathbf{\Sigma}_{\mathbf{NN}}$, because of the matrix product in Eq. 12.

The three actions, **Q**, $\mathbf{e}_0$, and $\mathbf{U}_0$ may be combined in a total action vector, $\mathbf{Q}_T$, defined by:

$$\mathbf{Q}_T \sim \left(\begin{Bmatrix} \mu_Q \\ \mu_\varepsilon \\ \mu_U \end{Bmatrix}, \begin{bmatrix} \mathbf{\Sigma}_{QQ} & \mathbf{\Sigma}_{Qe} & \mathbf{\Sigma}_{Qu} \\ \mathbf{\Sigma}_{Qe} & \mathbf{\Sigma}_{ee} & \mathbf{\Sigma}_{eu} \\ \mathbf{\Sigma}_{Qu} & \mathbf{\Sigma}_{eu} & \mathbf{\Sigma}_{uu} \end{bmatrix} \right) \tag{16}$$

The moments of $\mathbf{N} = [\mathbf{A}_q\, \mathbf{A}_e\, \mathbf{A}_u]\, \mathbf{Q}_T = \mathbf{A}_T\, \mathbf{Q}_T$ may be computed using Eqs. 11 and 12. This computation is simplified if it is assumed that $\mathbf{\Sigma}_{\mathbf{Qe}} = 0$, $\mathbf{\Sigma}_{\mathbf{Qu}} = 0$, and $\mathbf{\Sigma}_{\mathbf{eu}} = 0$; that is, if all pairs of actions from two different action vectors are assumed to be uncorrelated.

In lieu of modeling **Q** as a second moment vector, it may be modeled as a vector of interval numbers, with each component defined by upper and lower bounds. For example, for the loads on the Navier truss:

$$\mathbf{Q} = \begin{Bmatrix} [Q_1^l;\, Q_1^u] \\ [Q_2^l;\, Q_2^u] \end{Bmatrix} \tag{17}$$

In which Q_i^u and Q_i^l are the upper and lower bounds of Q_i. Any response vector of a linear elastic system is also a vector of interval numbers. For the Navier truss:

$$\begin{Bmatrix} \\ N_2 \\ \\ \end{Bmatrix} = \begin{Bmatrix} \left[\sum_{j=1}^{2} \min(a_{1j}Q_j^u, a_{1j}Q_j^l); \quad \sum_{j=1}^{2} \max(a_{1j}Q_j^u, a_{1j}Q_j^l) \right] \\ \left[\sum_{j=1}^{2} \min(a_{2j}Q_j^u, a_{2j}Q_j^l); \quad \sum_{j=1}^{2} \max(a_{2j}Q_j^u, a_{2j}Q_j^l) \right] \\ \left[\sum_{j=1}^{2} \min(a_{3j}Q_j^u, a_{3j}Q_j^l); \quad \sum_{j=1}^{2} \max(a_{3j}Q_j^u, a_{3j}Q_j^l) \right] \end{Bmatrix} \tag{18}$$

In which a_{ij} are the coefficients of the $\mathbf{A}_q$ matrix in Eq. 5c. Assuming the bounds of the interval numbers in **Q** are (these assumed bounds correspond to “mean ± two standard deviations” as given in Eq. 13):

$$\mathbf{Q} = \begin{Bmatrix} [6, & 14] \\ [20, & 80] \end{Bmatrix} \tag{19}$$

Then, the interval vector, **N,** is:

$$\mathbf{N} = \begin{Bmatrix} [7.026, & 23.89] \\ [13.686, & 52.344] \\ [8.287, & 60.766] \end{Bmatrix} \tag{20}$$

Q may also be modeled as a vector of fuzzy numbers. A fuzzy number may be viewed as a set of interval numbers, each with different bounds corresponding to a different "level of presumption", α. Figure 5d shows a "triangular" fuzzy number. The interval bounds associated with a specific level of presumption, α_1, are denoted as ${}_{\alpha 1}Q_i^l$ and ${}_{\alpha 1}Q_i^u$. If the load vector is modeled as a fuzzy vector, then any response vector will also be a fuzzy vector. For example, an axial force response of the Navier system at a specific level of presumption, α_1, denoted by ${}_{\alpha 1}N_i$, is an interval number with bounds:

$$[{}_{\alpha 1}N_i^l;\ {}_{\alpha 1}N_i^u] = \left[\sum_{j=1}^{2} \min(a_{ij}\,{}_{\alpha 1}Q_j^u, a_{ij}\,{}_{\alpha 1}Q_j^l);\quad \sum_{j=1}^{2} \max(a_{ij}\,{}_{\alpha 1}Q_j^u, a_{ij}\,{}_{\alpha 1}Q_j^l)\right] \tag{21}$$

If it is assumed that Q_1 and Q_2 are fuzzy numbers defined by figure 5a and 5b, then the element force N_3 is also a fuzzy number that may be computed by repeated use of Eq. 21 for various values of α. The result is shown in figure 5c.

Observations. A load vector, **Q**, may be modeled as a second moment random vector defined by its mean and covariance matrices. The covariance matrix of **Q** should be assembled from load data, which should include information on the linear correlation between load components. Mean and covariance matrices of

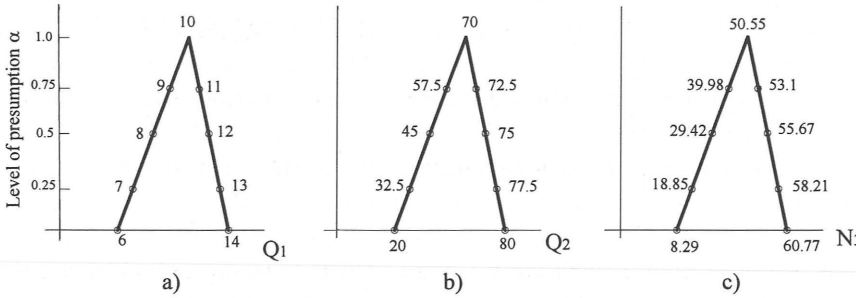


Figure 5
Fuzzy number models for Q_1 and Q_2; fuzzy number response, N_3

responses computed using Eqs. 12 and 13 may be considered to be conditional on the deterministic system model that is being used. Therefore the computed statistical moments of responses do not fully capture all of the possible response uncertainty. Nonetheless, response moments reflect the effects of correlation between loads and provide estimates of the coefficient of variation of all responses. The analyses do not predict extreme values of responses and their associated probabilities of nonexceedance, unless it is assumed that all random variables are normal.

An interval vector load model does not contain the concept of linear correlation between components. However, response bounds may be computed directly. The computation does not involve multiplication of interval numbers, only addition/subtraction and multiplication by constants are required. The arithmetic of adding interval bounds directly provides maximum and minimum responses from "pattern loads".

As for deterministic linear elastic analyses, nondeterministic linear elastic analyses do not predict strength and, for systems with unknown details or boundary conditions, they may be "non-predictive".

Linear elastic analyses. Nondeterministic systems and loads

Although a structural model may have been defined well and may be predictive of the behavior of an actual structure, the computed responses depend on parameters that may not be deterministic. For example, for basic linear elastic frame models, the modulus of elasticity or the element properties such as areas and moments of inertia may be uncertain. The nodal coordinates may also have some uncertainty, although usually it is not significant. Even if a model has been "refined", say by modeling connection behavior and support flexibility as shown in figure 3, the additional model parameters may also be uncertain. As for loads, system parameters may be modeled nondeterministically using random variables, interval numbers, or fuzzy numbers. Formulations that use random variables and random fields are usually called "stochastic finite elements" (Vanmarcke and Grigoriu 1983; Spanos and Ghanem 1989; Deodatis and Shinozuka 1991); formulations that use interval or fuzzy number models are called "interval finite elements" (Köylüoğlu and Elishakoff 1998) or "fuzzy finite elements" (Muhanna and Mullen 1999).

In stochastic finite element formulations the usual assumption made is that the nodal coordinates are deterministic and that the principal sources of uncertainty are the "EA" and "EI" terms in the stiffnesses. The additional considerations that arise from treating the EA and EI terms as random variables may be illustrated by considering the Navier truss. The system equilibrium equations in stiffness form are:

$$\begin{bmatrix} \frac{144}{169}K_1+\frac{1}{2}K_2+\frac{16}{25}K_3 & \frac{60}{169}K_1+\frac{1}{2}K_2-\frac{12}{25}K_3 \\ \frac{60}{169}K_1+\frac{1}{2}K_2-\frac{12}{25}K_3 & \frac{25}{169}K_1+\frac{1}{2}K_2+\frac{9}{25}K_3 \end{bmatrix}\begin{Bmatrix} u_l \\ v_l \end{Bmatrix} = \\ = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}\begin{Bmatrix} u_l \\ v_l \end{Bmatrix} = \mathbf{MQ} \tag{22}$$

In which $K_i = E_iA_i/L_i$ are the element axial stiffnesses. Terms of both the stiffness matrix and the load vector are now random variables. The element axial stiffnesses are now treated as random variables, which may be modeled as correlated second moment variables:

$$\begin{Bmatrix} K_1 \\ K_2 \\ K_3 \end{Bmatrix} \sim \left(\begin{Bmatrix} \mu_{K1} \\ \mu_{K2} \\ \mu_{K3} \end{Bmatrix}; \begin{bmatrix} \mathrm{var}[K_1] & \mathrm{cov}[K_1K_2] & \mathrm{cov}[K_1K_3] \\ \mathrm{cov}[K_2K_1] & \mathrm{var}[K_2] & \mathrm{cov}[K_2K_3] \\ \mathrm{cov}[K_3K_1] & \mathrm{cov}[K_3K_2] & \mathrm{var}[K_3] \end{bmatrix} \right) \tag{23}$$

The three *system* stiffnesses, written in vector form as $[K_{11}\ K_{12}\ K_{22}]^T$, are simply linear combinations or sums of the stiffness contributions of the elements:

$$\begin{Bmatrix} K_{11} \\ K_{12} \\ K_{22} \end{Bmatrix} = \begin{bmatrix} \frac{144}{169} & \frac{1}{2} & \frac{16}{25} \\ \frac{60}{169} & \frac{1}{2} & -\frac{12}{25} \\ \frac{25}{169} & \frac{1}{2} & \frac{9}{25} \end{bmatrix}\begin{Bmatrix} K_1 \\ K_2 \\ K_3 \end{Bmatrix} \tag{24}$$

Therefore the vector of stiffnesses that constitute the system stiffness matrix may be directly defined as a second moment vector using Eqs. 12 and 13. To compute moments of the displacement responses, the stiffness matrix may be inverted:

$$\begin{Bmatrix} u_l \\ v_l \end{Bmatrix} = \begin{bmatrix} \frac{1}{K_{11}-K^2_{12}/K_{22}} & \frac{1}{K_{12}-K_{11}K_{22}/K_{12}} \\ \frac{1}{K_{12}-K_{11}K_{22}/K_{12}} & \frac{1}{K_{22}-K^2_{12}/K_{11}} \end{bmatrix}\mathbf{MQ} = \\ = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}\mathbf{MQ} = \mathbf{K}^{-1}\mathbf{MQ} \tag{25}$$

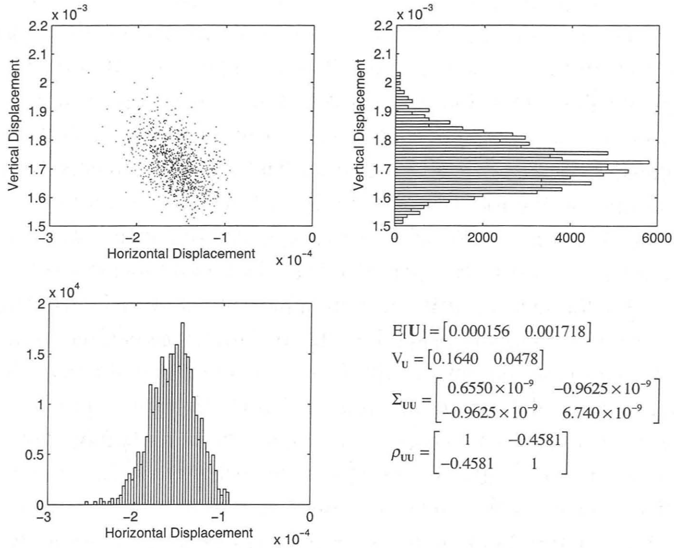
The *system* flexibilities, written in vector form as $[f_{11} f_{12} f_{22}]^T$, are also a second moment vector. Each flexibility is an inverse of a "reduced stiffness". The matrix inverse is a nonlinear operation; that is, the system flexibilities are not obtained by a linear transformation of the system stiffnesses. Therefore moments of the vector of system flexibilities must be estimated, typically by Taylor series expansions of the flexibilities about the mean values of the stiffnesses or by Monte Carlo simulation. Eq. 25 shows that the flexibilities are multiplicatively coupled with the load random variables. It is generally reasonable to assume that system stiffnesses or flexibilities are uncorrelated with the applied loads, **Q**. Therefore statistical moments of the nodal displacement vector may be computed using second moment algebra. The element forces may be expressed as:

$$\mathbf{N} = \begin{bmatrix} K_1 & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & K_3 \end{bmatrix} \begin{bmatrix} 12/13 & 5/13 \\ 1/\sqrt{2} & 1/\sqrt{2} \\ -4/5 & 3/5 \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \mathbf{MQ} \tag{26}$$

Therefore to compute statistical moments of the element force vector, **N**, the covariances between the element axial stiffnesses and the system flexibilities are needed. In lieu of these steps, Monte Carlo simulation may be used to estimate moments of the nodal displacements and element force responses. For the Navier truss, the vector of basic random variables is $[K_1\ K_2\ K_3\ Q_1\ Q_2]^T$. It may be defined as a second moment vector. For example, consider the following specific values:

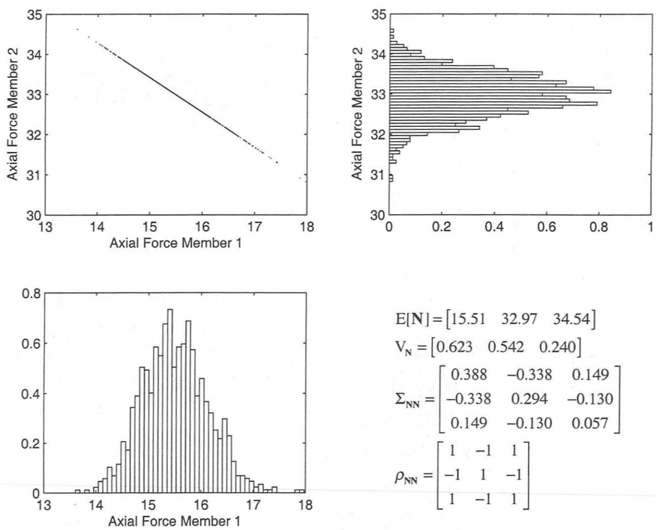
$$\begin{Bmatrix} K_1 \\ K_2 \\ K_3 \\ Q_1 \\ Q_2 \end{Bmatrix} \sim \left(\begin{Bmatrix} 30{,}000 \\ 30{,}000 \\ 30{,}000 \\ 10 \\ 50 \end{Bmatrix}; \begin{bmatrix} 2.5\times10^6 & 1.5\times10^6 & 1.5\times10^6 & 0 & 0 \\ 1.5\times10^6 & 2.5\times10^6 & 1.5\times10^6 & 0 & 0 \\ 1.5\times10^6 & 1.5\times10^6 & 2.5\times10^6 & 0 & 0 \\ 0 & 0 & 0 & 4 & 15 \\ 0 & 0 & 0 & 15 & 225 \end{bmatrix} \right) \tag{27}$$

The above assumed moments of $[K_1\ K_2\ K_3\ Q_1\ Q_2]^T$ imply that the loads are uncorrelated with the stiffnesses, that all stiffnesses have the same mean, variance, and coefficient of variation, 0.053, and that the stiffnesses are positively correlated with $\rho_{KiKj} = 0.6$. Monte Carlo simulation then involves assuming a density function for the random variables, generating realizations of correlated random variables whose moments match the prescribed mean and covariance matrices, performing linear elastic analyses, obtaining realizations of the responses, **U** and **N**, and computing statistical estimates of E[**U**], E[**N**], $\mathbf{\Sigma}_{\mathbf{UU}}$ and $\mathbf{\Sigma}_{\mathbf{NN}}$.



$$E[\mathbf{U}] = \begin{bmatrix} 0.000156 & 0.001718 \end{bmatrix}$$
$$V_U = \begin{bmatrix} 0.1640 & 0.0478 \end{bmatrix}$$
$$\Sigma_{UU} = \begin{bmatrix} 0.6550\times10^{-9} & -0.9625\times10^{-9} \\ -0.9625\times10^{-9} & 6.740\times10^{-9} \end{bmatrix}$$
$$\rho_{UU} = \begin{bmatrix} 1 & -0.4581 \\ -0.4581 & 1 \end{bmatrix}$$

Figure 6
Nodal displacements - nondeterministic system, deterministic loads

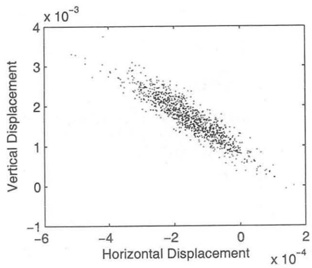


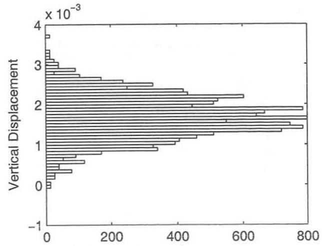
$$E[\mathbf{N}] = \begin{bmatrix} 15.51 & 32.97 & 34.54 \end{bmatrix}$$
$$V_N = \begin{bmatrix} 0.623 & 0.542 & 0.240 \end{bmatrix}$$
$$\Sigma_{NN} = \begin{bmatrix} 0.388 & -0.338 & 0.149 \\ -0.338 & 0.294 & -0.130 \\ 0.149 & -0.130 & 0.057 \end{bmatrix}$$
$$\rho_{NN} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

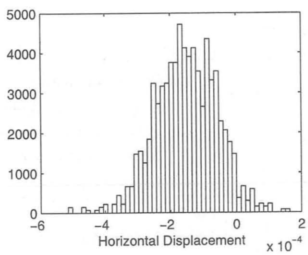
Figure 7
Element axial forces - nondeterministic system, deterministic loads

Simulations. Results of two Monte Carlo simulations are presented to illustrate effects of random system parameters on responses. One simulation assumes that the element stiffnesses are random but that the loads are deterministic, equal to their expected values. The other simulation assumes that both stiffnesses and loads are random, with moments given by Eq. 27. Both simulations assume Gaussian random variables; 1,000 realizations of the random vector were generated to estimate response moments. Figures 6 and 7 show scatter plots, histograms, means, and covariances of responses of the random system to deterministic loads. The responses may be compared with those given by Eqs. 14 and 15, which were computed analytically for the deterministic system subject to random loads. The expected displacements are essentially equal, but the coefficients of variation and the correlation coefficient between the two displacements are much smaller. The displacement histograms are skewed toward larger absolute values of displacements.

Therefore displacement responses of linear elastic, random (Gaussian) systems to deterministic loads are not Gaussian, because displacements are functions of the inverses of the random element stiffnesses. Figure 7 shows that expected values of the element forces are essentially the same as those of the deterministic system subject to random loads, but the coefficients of variation are very small. The element forces are perfectly correlated if the loads are deterministic.







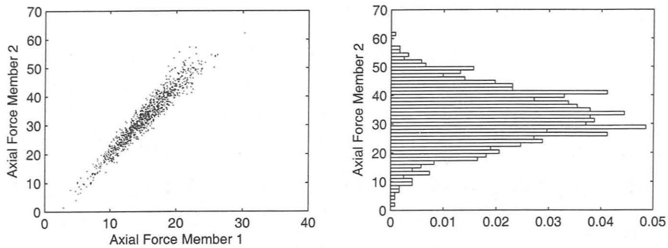
$$E[\mathbf{U}] = \begin{bmatrix} 0.000152 & 0.001687 \end{bmatrix}$$

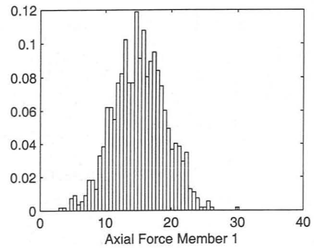
$$V_U = \begin{bmatrix} 0.6221 & 0.3235 \end{bmatrix}$$

$$\Sigma_{UU} = \begin{bmatrix} 8.955 \times 10^{-9} & -47.377 \times 10^{-9} \\ -47.377 \times 10^{-9} & 297.74 \times 10^{-9} \end{bmatrix}$$

$$\rho_{UU} = \begin{bmatrix} 1 & -0.9175 \\ -0.9175 & 1 \end{bmatrix}$$

Figure 8
Nodal displacements - nondeterministic system and loads





$$E[\mathbf{N}] = \begin{bmatrix} 15.22 & 32.45 & 33.84 \end{bmatrix}$$

$$V_N = \begin{bmatrix} 0.259 & 0.295 & 0.343 \end{bmatrix}$$

$$\Sigma_{NN} = \begin{bmatrix} 15.51 & 36.35 & 43.52 \\ 36.35 & 91.78 & 109.93 \\ 43.52 & 109.93 & 134.79 \end{bmatrix}$$

$$\rho_{NN} = \begin{bmatrix} 1 & 0.963 & 0.952 \\ 0.963 & 1 & 0.988 \\ 0.952 & 0.988 & 1 \end{bmatrix}$$

Figure 9
Element axial forces - nondeterministic system and loads

This result is specific to the case of deterministic loads on a node with three concurrent elements. The two nodal equilibrium equations constrain the magnitudes of the three element forces; given one element force, the other two must be deterministically related to it to satisfy equilibrium. If there were four or more elements converging to the node, the element forces would not be perfectly correlated. For a two-element, statically determinate truss, the element forces are determined by equilibrium. Therefore the element forces are deterministic and independent of the random element stiffnesses.

Figures 8 and 9 show corresponding responses for the case of random stiffnesses and loads. The expected displacements and forces are close to those of the deterministic system subject to random loads. The coefficients of variation are slightly larger and the correlation coefficients are slightly smaller. Although the system is nondeterministic, the displacement histograms are nearly symmetric because response variability arises largely from the random loads.

Observations. Any approximate analytical approach or simulation procedure for estimating responses of nondeterministic systems must satisfy the condition that element forces in statically determinate systems are independent of uncertainty in material and element properties. For hyperstatic systems, uncertainty in system parameters generally increases response coefficients of variation. Computed response moments of nondeterministic systems remain conditional on the defined system model and on the assumed load distribution.

Design based on nondeterministic linear elastic analyses

The text, *Structural and Civil Engineering Design* (Addis 1999) is an important compilation of articles on historical developments in structural design. The present view of the process is that a structural system is first conceptualized and then its acceptability is verified by iterative "detailed design." In current practice, detailed design involves defining load and system models and performing structural analyses to predict the effects of the loads. The effects are then compared with strength and serviceability design criteria. Strength is normally checked at the element level, using material and/or element strength models derived from testing and theory. These facets of detailed structural design developed gradually, starting with the theoretical advancements in linear elastic structural analysis by Navier (1826). Because of Navier's work, the dominant concept for achieving safety was to limit calculated stresses to fractions of measured material and element strengths. The work of Long (1836) and Mahan (1837) reflects Navier's influence in the U.S. The subsequent contributions of Rankine to such an "allowable stress design" (ASD) approach are discussed by Channell(1982) and Addis(1989–90). The limitations of ASD in turn motivated the development of "plastic design" in steel and "ultimate strength design" in concrete. These approaches achieve safety by using "load factors" and "resistance factors". Initially, single load factors were estimated by calibration with allowable stress design of statically determinate beams (Beedle 1957; Heyman 1957). However, it was understood that load factors also reflected uncertainty in the loads (Torroja 1958, ACI Committee 318 Report 1962). Therefore separate load factors for dead and live loads were introduced. For the recent development of the "load and resistance factor design" (LRFD) format in the U.S., it has been argued that the proposed factors correspond to target element reliabilities (Ravindra and Galambos 1978; Ellingwood et al. 1982). However, an LRFD format cannot assure uniform target element reliabilities because, with random loads, the variances of the load effects generally differ, as demonstrated by the nondeterministic analysis of the basic Navier truss. Although LRFD uses formulas that model element strength, factored load effects are usually computed using linear elastic analyses.

In 1967–69, Cornell suggested a probabilistic design format based on expected values and coefficients of variation of load effects and resistances (Cornell 1969). He did not advocate use of second-moment, nondeterministic linear elastic analyses to compute coefficients of variation of load effects. Rather, he suggested using coefficients of variation of load effects estimated from those of the "total load" and from "uncertainty in the structural analysis". Cornell's approach of explicitly estimating coefficients of variation of both resistances and

load effects is in fact what is needed to achieve uniform target element reliabilities. Considering the element strength design criterion, uniform target element reliabilities may be achieved by using a second moment reliability index, β (Cornell, 1969). For an element resistance, R, and a corresponding uncorrelated load effect, S, the reliability index is given by:

$$\beta = \frac{E[R] - E[S]}{\sqrt{\mathrm{var}[R] + \mathrm{var}[S]}} \tag{28a}$$

Or, introducing coefficients of variation, V_R and V_S,

$$\beta = \frac{E[R] - E[S]}{\sqrt{(V_R E[R])^2 + (V_S E[S])^2}} \tag{28b}$$

Eq. 28b is directly suitable for design. That is, E[S] and V_S may be computed using a second-moment nondeterministic linear elastic analysis, although such calculated V_S must be increased to capture all the uncertainties discussed by Torroja (1958) and Cornell(1969). V_R is estimated from statistical data on material strength and from uncertainties due to “fabrication” and “professional assumptions” (Cornell 1969). A target element reliability index, β, is prescribed. Then the required E[R] may be computed from (quadratic) Eq. 28b. Rather than using Eq. 28b directly, an equivalent alternate procedure may be used as follows. A “reduced” reliability index, β_S, is defined by introducing the “triangle inequality” approximation for the denominator in Eq. 28b:

$$\beta = \frac{E[R] - E[S]}{V_R E[R] + V_S E[S]} \tag{29}$$

$\beta_S \leq \beta$ because the denominator in Eq. 29 is an upper bound to the actual square root in Eq. 28b. Rearranging Eq. 29 and introducing the design inequality, yields:

$$E[R](1 - \beta_S V_R) \geq E[S](1 + \beta_S V_S) \tag{30}$$

The above equation, which was first derived by Lind (1971) using a different method, may be used to determine E[R]. The value of β_S that corresponds to a target element reliability, β, may be determined by substituting the expression for E[R] from Eq. 30 into Eq. 28b. The substitution leads to the following quadratic equation for β_S in terms of β and the coefficients of variation:

$$(2V_S^2 V_R^2 - (V_S + V_R)^2 / \beta^2)\,\beta_S^2 + 2V_S V_R (V_R - V_S)\,\beta_S + V_S^2 + V_R^2 = 0 \tag{31}$$

Therefore an equivalent alternate design procedure involves solving for β_S from the above equation and then determining E[R] from the simpler Eq. 30.

For example, the required expected values of the strengths of the three elements in the Navier truss may be determined as follows. From the simulation of the nondeterministic system, the coefficients of variation of the axial force responses are:

$$V_{N1} = 0.25885$$
$$V_{N2} = 0.29526$$
$$V_{N3} = 0.34304$$

Assume $V_R = 0.1$ and prescribe a target element reliability index $\beta = 4$. Then, from Eq. 31, the three β_S values are:

$$\beta_{N1} = 2.830$$
$$\beta_{N2} = 2.835$$
$$\beta_{N3} = 2.845$$

Using Eq. 30, the required expected values of strength are:

$$E[R_1] = 2.416E[N_1]$$
$$E[R_2] = 2.564E[N_2]$$
$$E[R_3] = 2.762E[N_3]$$

These expected element strength values provide uniform target element reliabilities for the computed coefficients of variation of responses. However, the system reliability is not explicitly quantified.

A similar approach may also be used for serviceability limit states such as, say, for displacements. "Limit displacements" should be prescribed in terms of expected values and coefficients of variation. Then, given the computed moments of displacement responses, second moment system (and element) *serviceability* reliability indices may be computed. If the serviceability reliability indices are smaller than the target values, the element stiffnesses must be increased.

Mechanism limit state analyses. Deterministic systems and loads

The design of a structure should consider its safety at all stages of its life: during construction, in normal usage, under extreme design events, under repeated cyclic actions, and in various stages of environmental degradation. An engineer must imagine potential failure modes and design to achieve an acceptable reliability in all potential failure scenarios. One potential limit state of a structure

occurs by static overloading, when a set of elements yield or rupture or a set of "plastic hinges" form such that a part or all of the structure becomes a mechanism, allowing large kinematic motions without increases in load. Controlling the likelihood of such a failure mode can form a basis for structural design, in addition to the criterion of assuring satisfactory performance during normal conditions.

How a structure reaches a failure mechanism limit state depends on the actual behavior of its components or members, which in turn depends on their stability and on material behavior. There are many linear or nonlinear, time-dependent or time-independent, one- or multi-dimensional material constitutive models. Time-independent, one-dimensional material models are sufficient for defining the static behavior of axial (truss) elements and plane frame beam elements. If the elements are stable, then their strength behavior is the same as that of the material. Figure 10 shows basic one-dimensional models for (stable) axial elements, which correspond to their one-dimensional stress-strain material models. In addition to the basic elastic-plastic and elastic-brittle models, Figures 10c and 10d show models for prestressed, tension-only (cable) or compression-only (contact) duc-

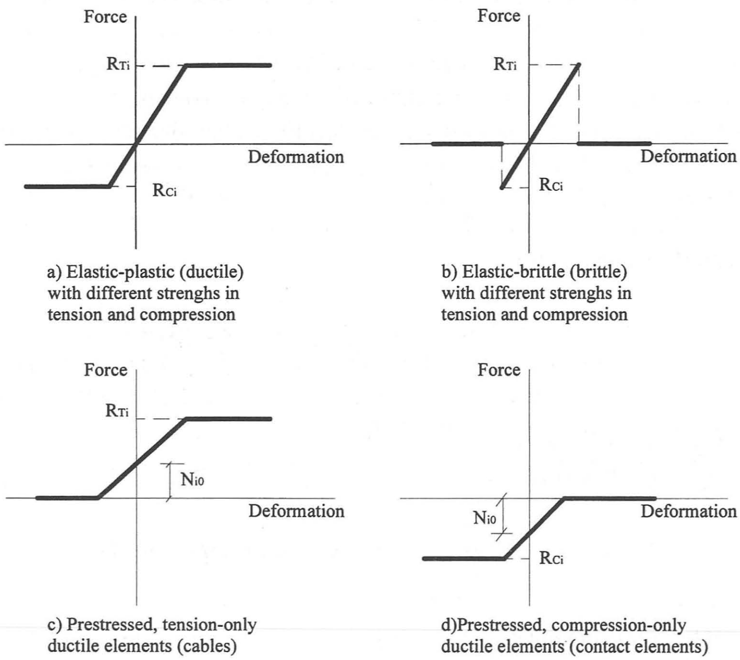


Figure 10
Force-deformation models

tile elements. The behavior of a structure is fundamentally different depending on whether the elements are ductile or brittle. If a brittle element breaks, its shed force must be redistributed to the remaining structure. If a ductile element yields, its stiffness becomes zero but it maintains its force through additional deformation. Analyses for predicting strength in these two conditions differ.

Ductile elements. Heyman (1998) provides a historical perspective on the development of procedures for predicting the strength of framed structures consisting of (stable) ductile elements. Such limit state analyses are usually conditional on a proportional loading. The objective is to determine the load factor corresponding to the unique collapse mechanism that satisfies equilibrium and the yield condition. Historically, two manual approaches, the mechanism method and the equilibrium method, evolved on the basis of the upper and lower bound theorems. Current structural analysis programs increment either a load or a displacement and trace the sequential development of element yielding or plastic hinge formation until the stiffness matrix becomes singular. Herein, for purposes of comparing deterministic and nondeterministic limit state analyses, the manual "mechanism method" is used. That is, all the possible failure mechanisms of the Navier truss are enumerated and the lowest load factor is associated with the unique controlling mechanism. The system consisting of elements with both tensile and compressive strengths is considered first. The set of equilibrium equations interpretable as limit state functions may be determined by prescribing compatible, virtual deformations and displacements that define a mechanism and then invoking the Principle of Virtual Displacements. The set of limit state functions for the Navier truss is as follows:

Elements 1 and 2 yield:

$$g_1 = \frac{14}{13} R_{T1} + \frac{7}{4\sqrt{2}} R_{T2} - \left(\frac{3}{4} Q_1 + Q_2 \right) \tag{32a}$$

Elements 1 and 3 yield:
If $-Q_1 + Q_2$ is positive

$$g_2 = \frac{7}{13} R_{C1} + \frac{7}{5} R_{T3} - (-Q_1 + Q_2) \tag{32b}$$

If $-Q_1 + Q_2$ is negative

$$g_3 = \frac{7}{13} R_{T1} + \frac{7}{5} R_{C3} + (-Q_1 + Q_2) \tag{32c}$$

Elements 2 and 3 yield:

If $-\frac{5}{12}Q_1 + Q_2$ is positive

$$g_4 = \frac{7}{12\sqrt{2}}R_{T2} + \frac{14}{15}R_{T3} - \left(-\frac{5}{12}Q_1 + Q_2\right) \tag{32d}$$

If $-\frac{5}{12}Q_1 + Q_2$ is negative (32e)

$$g_5 = \frac{7}{12\sqrt{2}}R_{C2} + \frac{14}{15}R_{C3} - \left(-\frac{5}{12}Q_1 + Q_2\right) \tag{32e}$$

For any prescribed proportion between Q_1 and Q_2, there are only three possible mechanisms. For example, consider the following deterministic values of load and strengths:

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{Bmatrix} 0.2 \\ 1.0 \end{Bmatrix} Q; \quad \begin{Bmatrix} R_{T1} \\ R_{T2} \\ R_{T3} \end{Bmatrix} = \begin{Bmatrix} 1.0 \\ 1.2 \\ 1.4 \end{Bmatrix} R_0; \quad \begin{Bmatrix} R_{C1} \\ R_{C2} \\ R_{C3} \end{Bmatrix} = \begin{Bmatrix} 0.5 \\ 0.6 \\ 0.7 \end{Bmatrix} R_0 \tag{33}$$

Then the three possible limit state functions are g_1, g_2, and g_4. The controlling mechanism corresponds to limit state function g_4; that is, elements 2 and 3 yield in tension at a critical load factor Q/R_o =1.965. If R_o = 80kN, then a mechanism will form when Q = 1.965(80) = 157 kN.

Now consider the Navier truss again, but as shown in figure 11, with two cable and one contact element, prestressed with initial forces N_{10}, N_{20}, and N_{30}. Because they are tension-only and compression-only elements, $R_{c1} = 0$, $R_{c3} = 0$, and $R_{t2} = 0$.

Substituting these zero strengths in Eqs. 32 gives the following modified limit state equations:

$$g_1 = \frac{14}{13}R_{T1} - \left(\frac{3}{4}Q_1 + Q_2\right) \tag{34a}$$

$$g_2 = \frac{7}{5}R_{T3} - \left(-Q_1 + Q_2\right) \tag{34b}$$

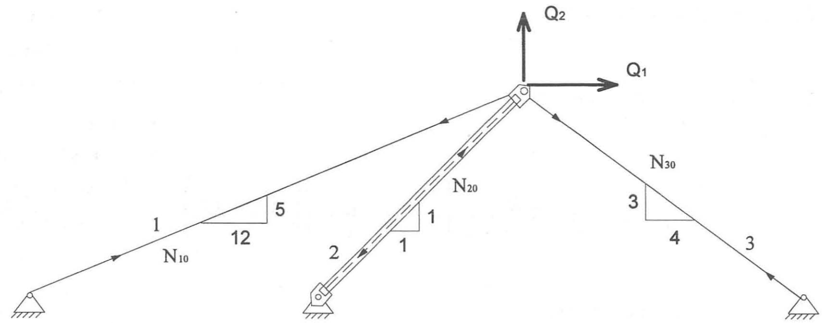


Figure 11
Navier truss with tension-only and compression-only elements

$$g_3 = \frac{7}{13} R_{T1} - \left(- Q_1 + Q_2 \right) \tag{34c}$$

$$g_4 = \frac{14}{15} R_{T3} - \left(- \frac{5}{12} Q_1 + Q_2 \right) \tag{34d}$$

$$g_5 = \frac{7}{12\sqrt{2}} R_{C2} - \left(- \frac{5}{12} Q_1 + Q_2 \right) \tag{34e}$$

Because tension-only and compression-only elements are used, only one element contributes to the internal virtual work when a mechanism is formed. With the assumed deterministic strengths and loads given by Eq. 33, the three appropriate limit state functions are g_1, g_2, and g_4. The controlling mechanism corresponds to g_1, which occurs at a load factor $Q/R_o = 0.936$, less than half of the load factor for the truss with elements having both tensile and compressive strengths.

Observations. In addition to the central concepts contained in the limit theorems and in the uniqueness theorem, deterministic limit state analyses provide other insights on behavior. For one, relative element stiffnesses and elastic distributions of forces do not affect strength. Nor is strength affected by an initial state of self-equilibrated forces. Therefore initial stresses from lack-of-fit or from small support movements do not affect strength. A system of ductile elements may be viewed as a set of "ductile failure modes in series". The strength of the system is controlled by the "weakest failure mode". Deterministic limit state analyses do not provide checks on performance at service loads. The analyses are conditional on a load proportion and on a set of deterministic strengths. They do not provide quantitative estimates of the probability of failure in a mechanism limit state mode.

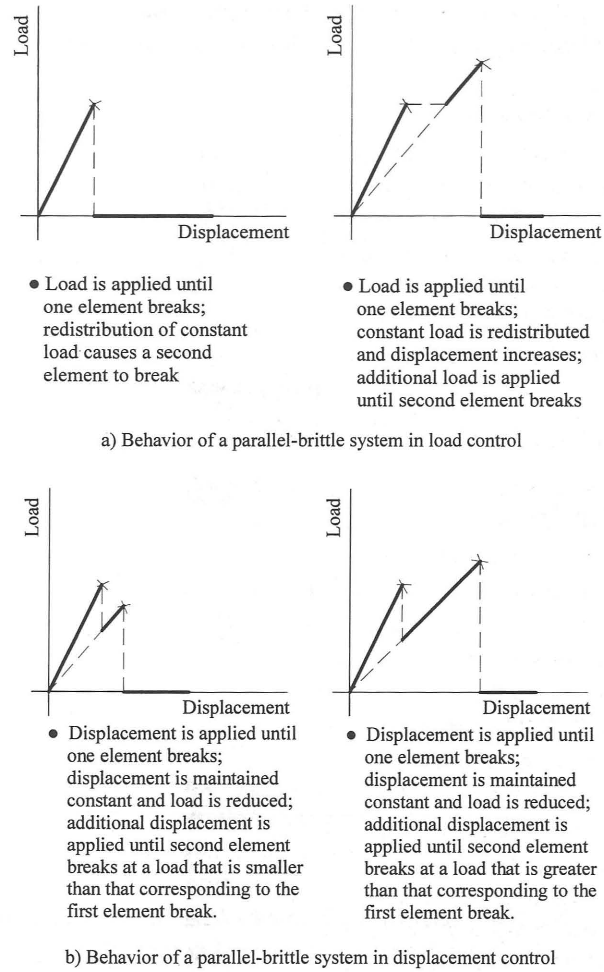


Figure 12
Load-displacement behavior of a parallel-brittle system

Brittle elements. Consider now the Navier truss with brittle elements as depicted in figure 10b. Four possible load displacement behaviors of the system under load and displacement control are shown in figure 12. In general, a hyperstatic brittle system may attain its maximum load or strength at the failure of the first, second, or n^{th} element. After failure of one or more elements, the system remains linear elastic, but with reduced stiffness. Consider the Navier

truss with the deterministic loads and strengths prescribed in Eq. 33. The forces in the elements in all possible states before a mechanism is formed are as follows:

All three elements active

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} = \begin{Bmatrix} 0.3091 \\ 0.6603 \\ 0.6903 \end{Bmatrix} Q \tag{35a}$$

Element 1 broken

$$\begin{Bmatrix} N_2 \\ N_3 \end{Bmatrix} = \begin{Bmatrix} 0.9293 \\ 0.5714 \end{Bmatrix} Q \tag{35b}$$

Element 2 broken

$$\begin{Bmatrix} N_1 \\ N_3 \end{Bmatrix} = \begin{Bmatrix} 1.0679 \\ 0.9821 \end{Bmatrix} Q \tag{35c}$$

Element 3 broken

$$\begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} = \begin{Bmatrix} -1.4857 \\ 2.2223 \end{Bmatrix} Q \tag{35d}$$

Setting $[N_1, N_2, N_3]^T$ equal to the element strengths, it is determined that as the load is increased element 2 breaks first at a load factor $Q/R_o = 1.817$. Considering the remaining $[N_1, N_3]^T$ forces under load control, it may be inferred that a second element breaks immediately and that a mechanism is formed. The strength of the brittle system is therefore $Q/R_o = 1.817$, a value that is smaller than that of the ductile system.

Consider next the brittle system with a set of initial, self-equilibrated forces as follows:

$$\begin{Bmatrix} N_{10} \\ N_{20} \\ N_{30} \end{Bmatrix} = \begin{Bmatrix} 0.5000 \\ -0.4350 \\ 0.1935 \end{Bmatrix} R_0 \tag{36}$$

Superposing forces in the initial condition with all elements active gives:

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} \begin{Bmatrix} 0.3091 \\ 0.6603 \\ 0.6903 \end{Bmatrix} Q + \begin{Bmatrix} 0.5000 \\ -0.4350 \\ 0.1935 \end{Bmatrix} R_0 \tag{37}$$

In this case, element 1 will break first at a load factor $Q/R_o = 1.6176$. Then considering equilibrium under load control with element 1 broken, element 2 will break immediately thereafter. Therefore the strength of the brittle system with the prescribed set of initial self-equilibrated forces is $Q/R_o = 1.6176$, lower than the strength of both the ductile system and of the brittle system with zero initial forces. Therefore an initial self-equilibrated state of stress can decrease the strength of a hyperstatic brittle system.

Mechanism limit state analyses. Nondeterministic strengths and loads

There are several conceptual differences between deterministic and nondeterministic limit state analyses of framed structures. If strengths and loads are considered to be random variables, there is no "proportional loading", no unique controlling mechanism, and no minimum load factor that defines the system strength. Rather, modal and system "reliability indices" or probabilities of failure are computed. As for the case of deterministic limit state analyses, nondeterministic limit state analyses differ depending on whether the elements are ductile or brittle.

*Ductile elements with both tensile and compressive strengths***.** The principal concepts of nondeterministic limit state analyses of ductile systems may be illustrated by again considering Navier's truss. In the context of nondeterministic analysis, the strength and load quantities in the limit state equations (Eqs. 32) are viewed as a vector of basic random variables, $\mathbf{X} = [R_{T1}, R_{c1}, R_{T2}, R_{c2}, R_{T3}, R_{c3}, Q_1, Q_2]^T$. A random vector may be completely defined by a multidimensional probability density function. Alternatively, the vector may be partially described by its mean and covariance matrices, $\mathbf{X} \sim (\boldsymbol{\mu}_X ; \boldsymbol{\Sigma}_{XX})$. Such a second moment partial description of the basic random variables allows probabilistic limit state analyses as outlined in figure 13. First, the vector $\mathbf{X}$ is transformed into standard space (variables with zero means, unit variances, and zero covariances) using the algorithm given by Rubinstein (1981). The hyperplane modal limit state equations, $\mathbf{g(X)}$, are also transformed into standard space and denoted as $\mathbf{g'(U)}$. A second-moment modal reliability index, β_i, is defined for each mode $g_i'(\mathbf{U})$ (Cornell 1969; Veneziano 1974). Figure 13 shows the meaning of β_i in two-dimensional space; that is, β_i is the (minimum) distance from the origin in standard space to the limit state hyper-

plane, $g_i'(\mathbf{U})$. The point on $g_i'(\mathbf{U})$ closest to the origin, denoted by $\mathbf{U}_i^*$, is called a "design point". β_i is a scalar second moment index of modal reliability. Recognizing that the structure may be viewed as a set of ductile modes in series, the system is safe only if none of the failure modes occur. Hasofer and Lind (1974) proposed the minimum of the modal β_i's, denoted as β_{HL}, as a scalar second moment index of system reliability. For hyperplane modal limit state functions, the modal reliability indices, β_i, may be computed analytically. As an example, assume $\mathbf{X}$ to have the following moments:

$$\mathbf{X}=\begin{Bmatrix} R_{T1}\\ R_{C1}\\ R_{T2}\\ R_{C2}\\ R_{T3}\\ R_{C3}\\ Q_1\\ Q_2 \end{Bmatrix} \sim \left(\begin{Bmatrix} 80\\ 40\\ 96\\ 48\\ 112\\ 56\\ 10\\ 50 \end{Bmatrix};\begin{bmatrix} 64 & 25.6 & 38.4 & 19.2 & 44.8 & 22.4 & 0 & 0\\ 25.6 & 16 & 19.2 & 9.6 & 22.4 & 11.2 & 0 & 0\\ 38.4 & 19.2 & 92.16 & 36.86 & 53.76 & 26.88 & 0 & 0\\ 19.2 & 9.6 & 36.86 & 23.04 & 26.88 & 13.44 & 0 & 0\\ 44.8 & 22.4 & 53.76 & 26.88 & 125.44 & 50.176 & 0 & 0\\ 22.4 & 11.2 & 26.88 & 13.44 & 50.176 & 31.36 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 15\\ 0 & 0 & 0 & 0 & 0 & 0 & 15 & 225 \end{bmatrix}\right) \tag{38}$$

The form of $\mathbf{\Sigma_{XX}}$ implies that the loads are assumed to be uncorrelated with the strengths. The variances of the strengths were computed assuming C.O.V. $[R_i]$ = 0.1. The covariances between any two strengths were computed assuming $\rho_{RTiRTj} = 0.5$, $\rho_{RTiRcj} = 0.5$, $\rho_{RciRcj} = 0.5$, and $\rho_{RTiRci} = 0.8$. For the loads, it was assumed that C.O.V.$[Q_1]$ = 0.2, C.O.V.$[Q_2]$ = 0.3, and $\rho_{Q1Q2} = 0.5$. Following the steps shown in Figure 13, the second moment modal reliability indices were computed to be: $\boldsymbol{\beta}^T$ = [6.19, 6.29, 9.13, 5.05, 7.38]. The Hasofer-Lind system reliability index is: $\beta_{HL} = \beta_{min} = \beta_4 = 5.05$. The "most likely" mode is 4, which is also the controlling mode from the deterministic limit state analysis. The meaning of $\beta_{HL} = 5.05$ in terms of system reliability may be determined by estimating the system reliability numerically, say by simulation. Crude or "brute force" Monte Carlo simulation involves assuming a probability density function for the variables, generating samples of the random vector, $\mathbf{U}$, and computing the fraction of the total number of realizations that occur "outside" the convex limit state surface of the system. When probabilities of failure are small, it is necessary to use "variance reduction" statistical techniques such as "importance sampling" (Ross 1990) in order to decrease the number of realizations required for a good estimate of system reliability. Assuming normal random variates and using Monte Carlo simulation with importance sampling, it is estimated that P[failure] $\tilde{}\ 2.26 \times 10^{-7}$. Denoting $\Phi(u)$ as the cumulative distribution function for a standard normal variate, it can be noted that $1 - \Phi(5.05)$, is approximately 2×10^{-7}, which

- Second moment vector of basic random variables

  Real space: $\mathbf{X} \sim (\boldsymbol{\mu}_{\mathbf{X}}; \boldsymbol{\Sigma}_{\mathbf{XX}})$

  Standard space: $\mathbf{U} \sim (\mathbf{0}, \mathbf{I})$
- Hyperplane modal limit state static equations that define a convex failure surface

  $\mathbf{g}(\mathbf{X}) = \mathbf{AX}$ $\qquad$ $\mathbf{g}'(\mathbf{U}) = \mathbf{BU} + \mathbf{C}$
- Scalar hyperplane modal limit state equation

  $g_i(\mathbf{X}) = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n$ $\qquad$ $g_i(\mathbf{U}) = b_{i1}u_1 + b_{i2}u_2 + \cdots + b_{in}u_n + c_i$

  $g_i(\mathbf{X}) = \mathbf{a}_i^T\mathbf{X}$ $\qquad$ $g_i(\mathbf{U}) = \mathbf{b}_i^T\mathbf{U} + c_i$
- Scalar, second moment, modal reliability index, $\beta_i = \dfrac{c_i}{\sqrt{\sum_{j=1}^{n} b_{ij}^2}}$

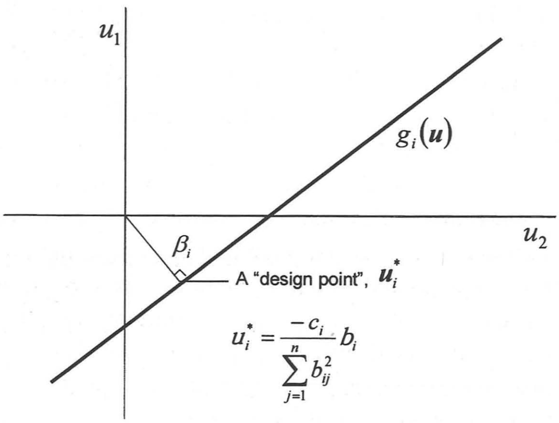


Figure 13
Second moment reliability analysis

is close to the estimated probability of system failure. This indicates that mode 4 "contributes the most" to the probability of system failure and that

$\beta_{HL} = \beta_{min} = \beta_4 = 5.05$ is a good indicator of system reliability. If several modes had β_i values slightly larger than 5.05, then the probability of system failure would increase whereas β_{HL} would not. Basically, β_{HL} relates system reliability to the reliability of the weakest mode.

Many studies may be performed by changing the moments of **X** and noting changes in the second moment modal reliability indices and in the probabilities of system failure. For example, consider the following new moments of **X**:

$$\mathbf{X} \sim \left\{ \begin{Bmatrix} 60 \\ 30 \\ 60 \\ 30 \\ 70 \\ 35 \\ 10 \\ 50 \end{Bmatrix}; \begin{bmatrix} 36 & 14.41 & 18 & 9 & 21 & 10.5 & 0 & 0 \\ 14.41 & 9 & 9 & 4.5 & 10.5 & 5.25 & 0 & 0 \\ 18 & 9 & 36 & 14.4 & 21 & 10.5 & 0 & 0 \\ 9 & 4.5 & 14.4 & 9 & 10.5 & 5.25 & 0 & 0 \\ 21 & 10.5 & 21 & 10.5 & 49 & 19.6 & 0 & 0 \\ 10.5 & 5.25 & 10.5 & 5.25 & 19.6 & 12.25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 15 \\ 0 & 0 & 0 & 0 & 0 & 0 & 15 & 225 \end{bmatrix} \right\} \tag{39}$$

In which the expected values of the resistances have been reduced. It is now found that $\boldsymbol{\beta}^T$ = [4.092 4.187 7.682 2.652 5.998]. Therefore the most likely mode remains mode 4 but now $\beta_{HL} = \beta_{min} = \beta_4 = 2.652$. Assuming normal random variates, the probability of failure estimated by Monte Carlo simulation is: P[failure] = 0.004. This value is again close to $1 - \Phi(2.652)$, the probability of failure of mode 4. Assuming *lognormal* random variables having the same mean and covariance matrices, it is estimated by Monte Carlo simulation that P[failure] = 0.0127. Therefore the Hasofer-Lind second moment system reliability index, β_{HL}, should not be associated with a unique probability of system failure.

It is of interest to study the effects of correlation between element resistances on system reliability. Assuming $\rho = 0.999$ between all pairs of (normal) resistances, it is estimated by Monte Carlo simulation that P[failure] = 0.005. Assuming $\rho = 0.0$ between all pairs of resistances, it is estimated that P[failure] = 0.0031. Therefore system reliability decreases as the ductile resistances become more correlated. Although the system may be viewed as a series combination of modes, each mode requires parallel yielding of elements. It is the reliability of a mode consisting of ductile elements in parallel that decreases as the correlation between resistances increases. And this decrease in modal reliabilities decreases the system reliability.

Ductile elements with only tensile or compressive strengths. Consider now the nondeterministic limit state analysis of the Navier truss with tension-only and compression-only elements as shown in figure 11. That is, the elements have deterministic zero strengths: $R_{c1} = R_{c3} = R_{T2} = 0.0$. For such a structure, there are five basic random variables: $\mathbf{X}^T = [R_{T1}\ R_{c2}\ R_{T3}\ Q_1\ Q_2]$. The moments given by Eq. 38 reduce to:

$$X \sim \left(\begin{Bmatrix} 80 \\ 48 \\ 112 \\ 10 \\ 50 \end{Bmatrix}; \begin{bmatrix} 64 & 19.2 & 44.8 & 0 & 0 \\ 19.2 & 23.04 & 26.88 & 0 & 0 \\ 44.8 & 26.88 & 125.44 & 0 & 0 \\ 0 & 0 & 0 & 4 & 15 \\ 0 & 0 & 0 & 15 & 225 \end{bmatrix} \right) \tag{40}$$

A second moment reliability analysis gives: $\boldsymbol{\beta}^T = [1.59\ 5.54\ 5.63\ 3.27\ 4.45]$. Therefore $\beta_{HL} = \beta_{min} = \beta_1 = 1.59$. Mode 1, which was the controlling mode in the corresponding deterministic limit state analysis, is now the most likely mode. The second moment system reliability index has decreased from 5.05 to 1.59 with the use of tension-only and compression-only elements. Using the smaller mean resistances given in Eq. 39, modified for the zero strengths, yields the following modal reliability indices:

$\boldsymbol{\beta}^T = [0.417\ 3.38\ 4.998\ 1.22\ 3.97]$. Assuming normal random variables, it is estimated by Monte Carlo simulation that P[failure] = 0.340. Therefore the use of tension-only and compression-only elements increased the estimated probability of failure from 0.004 to 0.34.

Brittle elements. In general, simulation is required to estimate the reliability of parallel-brittle systems. For each realization of the basic random variables, an incremental structural analysis with element deletion and force redistribution must be performed to determine if system failure occurs. The system reliability is estimated from the fraction of realizations that do not lead to failure. For the basic Navier truss, the set of (four) mutually exclusive survival modes may be enumerated: it can survive with all elements intact, or with only elements 1 and 2 intact, or with only elements 1 and 3 intact, or with only elements 2 and 3 intact. The probability of survival of any one mode may be expressed in terms of probability inequalities that follow from structural analysis of the system in the damaged condition associated with the particular survival mode. For example, the probability of survival with only elements 2 and 3 intact may be expressed as:

P[system survival with only elements 2 and 3 intact] = P[Failure element 1 in tension ∩ survival of element 2 in tension n survival of element 3 in both tension and compression] =

$$P\Bigg[R_{T1} - (0.421Q_1 + 0.225Q_2) \le 0 \cap R_{T2} - \left(\frac{3\sqrt{2}}{7}Q_1 + \frac{4\sqrt{2}}{7}Q_2\right) > 0$$
$$\cap R_{T3} - \left(-\frac{5}{7}Q_1 + \frac{5}{7}Q_2\right) > 0 \cap R_{C3} + \left(-\frac{5}{7}Q_1 + \frac{5}{7}Q_2\right) > 0\Bigg] \tag{41}$$

The probability of system survival in the other modes may be expressed in a similar way.

Because the modes are mutually exclusive, the probability of system survival is the sum of the probabilities of survival in each mode. In a Monte Carlo simulation, for each realization of the vector of basic variables, the inequalities associated with all four modes are checked to determine if the system survives. The fraction

of realizations in which the system fails provides an estimate of the probability of failure. Simulations were performed for the parallel-brittle Navier truss with normal basic random variables with means and variances given by Eq. 39 and different assumptions on the correlations between element resistances. Table 1 compares the estimated probabilities of failure of the parallel-brittle system with those of the parallel-ductile system. The reliabilities of the parallel-ductile and parallel-brittle systems approach one another as the correlations between element resistances increase, but the parallel-ductile system remains more reliable.

Correlation between resistances	*Estimated probability of failure of a parallel-brittle system*	*Estimated probability of failure of a parallel-ductile system*
0.0	0.0107	0.0031
0.5, 0.8	0.0097	0.004
0.999	0.008	0.005

Table 1
Estimated probabilities of failure of parallel-brittle and parallel-ductile systems

Strength design of ductile systems based on mechanism limit state analyses

With deterministic mechanism limit state analyses, the principal design objective is to provide element resistances such that collapse loads are greater than factored design loads. For example, the Navier truss collapse load, $Q = 1.965R_o$, should be greater than the factored design load. Other than increasing individual element resistances to obtain collapse loads greater than factored loads, there is no one, commonly-used, "detailed design" method that "optimizes" a design based on deterministic mechanism limit state analyses. A general objective of detailed design is to preclude "local" mechanisms that involve only one or a few elements. Heyman (1951) and Zeman and Irvine (1986) have defined algorithms to achieve minimum-weight optimal designs based on deterministic mechanism limit state analyses.

With nondeterministic mechanism limit state analyses, the principal design objective is to provide (expected values of) resistances such that the system reliability is acceptable. For small structures such as Navier's three-bar truss, it is possible to enumerate all failure modes, write explicit limit state functions, and determine the most likely failure mode. However, for most realistic structures, it is not possible to enumerate all the potential failure modes and write explicit limit state functions. In general, extensive simulations involving multiple incremental

limit-state analyses are needed to estimate reliability. A general detailed design process for optimally modifying individual elements to affect system reliability remains to be defined. Research on such "reliability-based optimal design" methods is ongoing (Frangopol 1997).

Summary and observations

The development of analysis-based design in the early 19th century revealed the uncertainties in modeling systems and live loads as well as the need for a method to achieve reliability, for both strength and serviceability. Discussion of these issues was largely quieted by the successful adoption of deterministic linear elastic analyses together with very conservative "working" live loads and allowable fractions of conservative material/element strengths. Deterministic linear elastic analyses have provided invaluable insights on structural behavior and have served as bases for the safe development of innovative structural forms. A very useful property of linearity is that any load effect vector may be expressed as a linear combination of the magnitudes of actions such as applied loads and prescribed support displacements. Although deterministic linear elastic analyses provide only "point estimates" of responses, the associated allowable stress design methods have produced safe designs.

The limitations of linear elastic analysis methods motivated the development of mechanism limit state analyses. The limit theorems, the uniqueness theorem, and the fact that an initial self-equilibrated state of stress does not affect the strength of systems of ductile elements are invaluable insights on structural behavior. With the introduction of "load factors", design methods based on deterministic limit state analyses have also provided safe designs for strength. Such design methods, however, must generally include separate checks on serviceability limit states.

Deterministic linear elastic and mechanism limit state analyses do not explicitly consider uncertainty in loads or in the material and element properties used in system models. Therefore element and system reliabilities are unknown, except that the historic reliabilities of systems designed by deterministic methods have, in general, been accepted. However, a better understanding of the factors that control reliability can improve designs. And it is rational to develop design methods that can provide uniform element reliabilities and quantifiable system reliability. It is these objectives that have motivated research on non-deterministic structural analysis and design methods.

Nondeterministic linear elastic analyses have been formulated by defining quantities as random variables, interval numbers, and fuzzy numbers. The most advanced development is associated with random-variable-based models; more specifically, with "second moment" models that use mean and covariance matri-

ces to define random variables. The complexity of nondeterministic linear elastic models depends on whether only the loads, or only the system, or both the loads and the system are modeled using random variables. If the system is considered deterministic, then the computation of the mean and covariance matrices of any response vector is straightforward. If both the system and loads are modeled using random variables, then some analytical approximations and/or Monte Carlo simulation is required to estimate mean and covariance matrices of responses. In both cases, the "radical" departure is that the loads are defined by mean and covariance matrices rather that "working" values.

Mean and covariance matrices of responses provide useful design information. A variance is a measure of the uncertainty in a response and a covariance is a measure of the linear correlation between two responses. The correlation between two responses is relevant to the design of an element for two or more load effects such as axial force and bending moment. The correlation indicates the likelihood that a pair of load effects may both have high values. Cornell (1969) proposed a design method based on estimating means and coefficients of variation of load effects and element resistances. The method provides for the design of elements with uniform target second moment reliability indices for strength. The method may also be used to attain a prescribed system reliability index for serviceability.

Nondeterministic mechanism limit state analyses do not use the concept of "proportional loading". There is no unique controlling mechanism and minimum load factor. Rather, a structure is viewed as a series system of failure modes. Depending on element behavior, a failure mode can in turn be considered as a parallel-ductile or parallel-brittle subsystem. Nondeterministic mechanism limit state analyses estimate reliability indices or probabilities of failure for the modes of a system. Nondeterministic limit state analyses quantify the effects of element behavior (ductile or brittle), element strength distribution, and correlations between random variables on modal and system reliability. In general, it is not possible to enumerate all failure modes. Thus failure surfaces have to be estimated and extensive simulation is required.

In summary, nondeterministic models may be said to be "more rational" in the sense that they model some of the real uncertainties in loads and systems. There are practical, well-developed, random-variable-based analysis/design methods, at least for linear elastic systems. These analysis/design methods provide useful insights on structural behavior. However, the impact of such procedures on the reliability and economy of designs must still be determined. In addition, for implementation, considerable change must occur in current analysis/design practice. Mean and covariance matrices of actions, system properties, and element resistances must be defined. Nondeterministic analysis

algorithms must be implemented in commonly-used finite element programs. Nondeterministic design procedures must be codified and accepted by legal entities. Most importantly, practicing design engineers must understand nondeterministic methods and be convinced of the merits of applying them.

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New lamps for old. Should computers provide engineers with answers or help us think?

Bill Harvey

The basic engineering skill of understanding structural behaviour is being lost through the growing belief that what the computer tells you is what actually happens. Jacques Heyman has trodden a quite lonely path arguing the case that we depend absolutely on the plastic theorems in design and neglect that understanding at our peril. The paper considers some aspects of masonry structures which do not lend themselves to what we see as "analysis" and considers the alternative of exploration of potential behaviour which is readily tackled with the help of a computer. In the old days (my youth!) the slide rule was the lamp but our own brain was the genie in engineering. We were sold a new lamp and foolishly believed we had a replacement Genie too. If we don't re-engage with the old one, the cost will be terrifying.

Introduction

Flexibility and Stiffness the thoughtful and the automatic

In the development of the analysis of hyperstatic structures through the 20th Century, two processes presented mirror image solutions based on the same view of behaviour. The more popular approach in the early stages involved releasing the structure until it became determinate, then computing the forces that would be required to restore the releases. One advantage of this process was that the engineer could, by careful choice of releases, minimise the computational effort and deliver an inherently stable mathematical solution. The key to the demise of the process is in that element of choice.

Once computers became available, there was a pressing need for an automatic process. The computer could not sit and wait while the engineer made choices, nor could it sensibly communicate what the choices might be and how they could be made. It was vital that the process be completely automatic.

The stiffness approach, in which displacements are the unknowns, solved this problem, while introducing a number of others. The process in stiffness solutions is to fix every potential movement and work out the forces that are generated by that fixity then consider the effect of releasing the joints.

The limits of knowledge and understanding

When an engineer designs a structure, the analytical element of his work is really quite straightforward. The loads to be supported, the geometry of the structure and materials from which it is made are known, in the sense that they are defined by the designer. Provided equilibrium is satisfied, plasticity protects the engineer from failure.

In assessment, the simplifications are no longer available. The geometry may be apparently simple but is often complex and difficult to discover. The materials may vary considerably from place to place and the engineer can have no idea what controls were exercised on workmanship. Past and even present loads are quite unknown.

In all structures, the boundary conditions have a considerable effect on behaviour. Those conditions will remain impossible to know. Again, the plasticity inherent in modern structures protects the engineer from failure in new design. In assessing existing structures, the result of all these problems is likely to be that the analysis predicts failure. Unlike the designer, the assessing engineer cannot change his structure to deal with that.

In assessment, analysis as a deterministic process becomes irrelevant. What the engineer must do is to explore the possibilities of behaviour rather than determine what actually happens. Most modern analytical tools do not lend themselves to such exploration. Dramatic increases in speed of computing are immediately gobbled up with further facilities and better visualisation. The author believes that this is a tragic waste of the wonderful tools available to us. Engineering programmes should be designed as levers, not replacements, for the mind. Such developments are unlikely to take place in an economy driven by market forces. Vendors of existing software, always have the power to prevent the development of a replacement. This may be by preventing a flow of funding or by simply buying and suppressing the development.

The object of this paper is to present an approach to analysing some particular structures using simple programming tools to develop effective exploratory models.

Arches and arching structures

Arches present a classic example of all the problems listed above. Exposed masonry arches, only really occur in flying buttresses. Even there we have no way of knowing the sequence of construction and release of formwork. The actual location of thrust in the buttress is controlled by the initial stress state, by movement of the foundations, by thermal effects, by wind and many other environmental changes. Elastic, and even elasto-plastic analysis is therefore of no value. Heyman discussed this in some detail, in his books "The Masonry Arch" (Heyman 1981) and "The Stone Skeleton" (Heyman 1995) and in many papers, most recently in The Structural Engineer (Heyman 2005). Throughout his work, he concentrated on hand calculation. The aim of this paper is to show that the computer by adding speed, visualisation and interaction to the simple hand calculation processes can deliver a new level of understanding to the engineer. The work also leans heavily on ideas presented by WH Barlow in a very early paper to the institution of Civil Engineers (Barlow 1846).

Arch bridges and masonry tunnels in soft ground.

The author's everyday work includes assessment of masonry arch bridges, tunnels and vaulted structures, carrying heavy loads. Each of these offers its own complexities.

The needs of the engineer

All scientists crave knowledge. Engineers are scientists but many other things besides. The artefact must be created, maintained even sometimes destroyed within the limits of knowledge available. Often, qualitative understanding is sufficient where quantitative knowledge is not possible. In the limit, each step in engineering is an issue of confidence. Ted Happold used to speak of the need to achieve the confidence to build, in dealing with existing structures this must be translated into the confidence to leave well alone.

Satisfying the needs

When faced with a structure where he cannot hope to know what is actually happening, the engineer must explore many possibilities. If simple hand calculations are sufficient the exploration might be carried out over a few hours or days. Unfortunately, that is rarely possible.

What becomes necessary is a responsive system, where the calculations are well understood, the output can be easily visualised and input controlled by

some form of direct interaction. Once properly designed software is in place, the engineer can explore many thousands of alternatives in a short period of time.

Exploration

An exploratory approach to analysis requires a point of input. With thrust based analysis it is easy to think in terms of a flexibility approach. The structure is rendered determinate by introducing releases. Instead of solving computationally for the restoring forces, they are treated as input to an interactive system. This creates a requirement for interactive tools. Luckily these are readily available within high-level programming environments such as Excel.

Visualisation

The basic calculations for arching structures are usually thrust based or equilibrium studies. The model must be formulated carefully to allow rapid response. It must also be designed from the start to allow easy visualisation of the output. Working in two dimensions thrust lines are quite straightforward but they don't tell a complete story. The author extended the concept to the zone of thrust, which contains enough material to support the force. This tells a more complete story but may still not be sufficient. Occasionally, it is necessary to develop new visualisations for a specific project.

The plotting functions within Excel, are extremely powerful, though it is sometimes necessary to give considerable thought to their application. One very powerful tool is indirect addressing, which allows numbers from various places to be gathered together for graphing.

Interaction

Having set up a computational scheme to deal with the basic calculation, visualisation and exploration, it remains to build the interactive framework. Working within Excel interaction tools appear either in the Forms Toolbox or in the Visual Basic Toolbox. The most useful tools are drop-down menus, radio buttons, check boxes, spin buttons and scroll bars. These can be dragged into position on the page and linked to any required cell.

Some examples

The processes discussed can only really be understood when examples are presented. Three will be presented here, showing different levels of complexity.

Viaducts

It is traditional to treat arch bridges as two-dimensional structures viewing only the elevation. The third dimension is dealt with at the input stage by distributing the load onto an effective strip. There is considerable doubt about the validity of this process over the length of a viaduct, but it does allow some understanding to be developed.

Each arch is divided into a number of blocks. The block carries its own weight, and forces from the fill above. The fill forces include loads transmitted from the live loads.

The minimum thrust is first calculated. That means that the thrust touches the intrados near the two springings and the extrados near the crown. The total force at the head of each pier is calculated from the two arch thrusts and the weight of material between the Arches. In flexibility terms, each arch is released with three hinges.

When live load is applied to one arch, thrust in the adjacent piers is displaced, often dramatically. This implies tilting of the pier, away from the loaded span. The adjacent spans resist this tilt by increasing compression. The hinge at the crown, moves down and those at the springings move up. The thrust may also skew slightly, rising at the loaded end and falling at the unloaded. The illustrations in figure 1 and figure 2 come not from an Excel spreadsheet but from a specialist program, Archie-M. They show the process of balancing thrusts that the engineer must go through during analysis.

Vaults

It may seem a natural step from two-dimensional structures to three-dimensional, but it is not so easy. The visualisation issues here are considerable, not simply in visualising the output from the calculations, but more particularly, visualising the way the calculations may be carried out. The author was faced with the need to assess a system of vaults carrying a railway station. The vaults were 10 metres span in each direction but were skewed at 18°. The piers were rhomboid with two metres sides. The vaults were required to carry railway vehicles with axle loads of up to 25 tonnes. The tracks made a complex pattern over the vaults.

The first step in understanding was to realise that the crowns of the vaults formed an intersecting grid of substantial horizontal members. These were amply able to transmit large horizontal forces to the abutments. This meant that each web of the vaults, could be treated independently and stabilised by the horizontal forces. A group of four webs was then gathered onto one pier and the thrust traced down the pier.

Figure 3 shows the page dealing with input interaction and visualisation. Each web was divided into sections of varying width. Three components of reaction at the

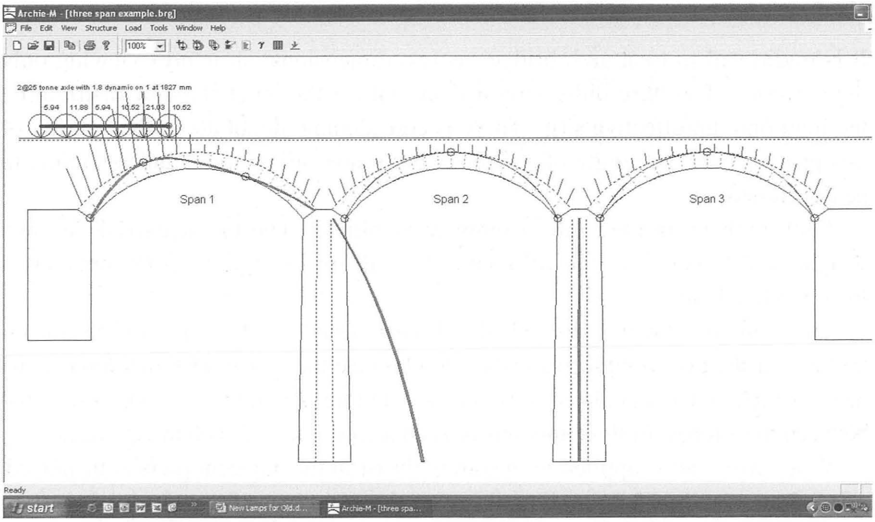

Figure 1
A three span bridge with minimum thrust in each arch

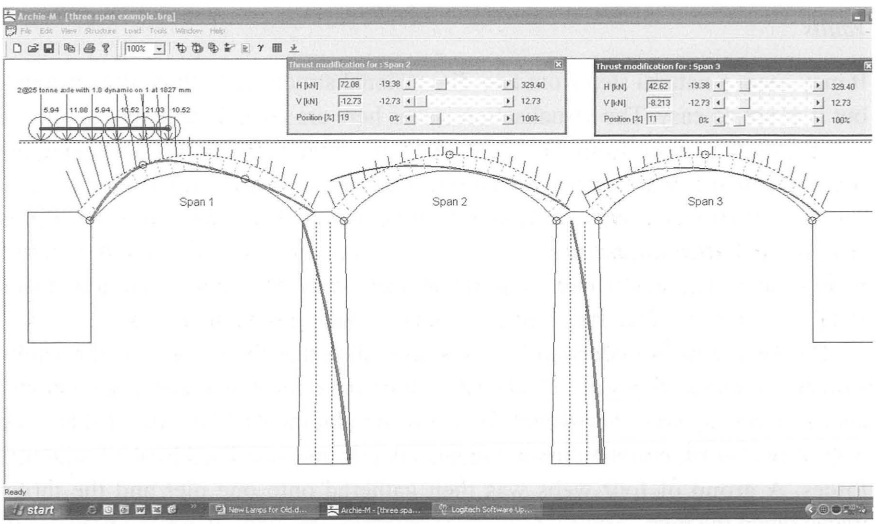

Figure 2
The bridge in figure 1 with thrusts adjusted

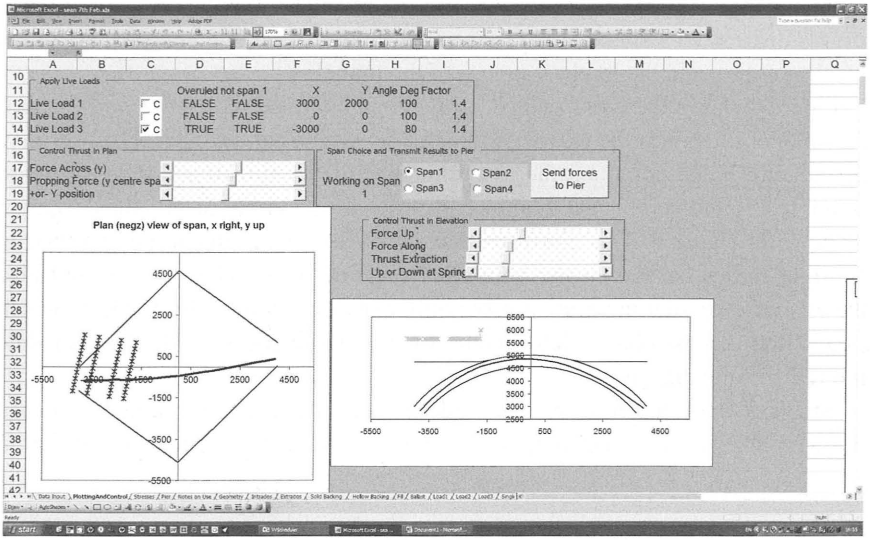

Figure 3

A spreadsheet solution for balancing thrusts in a system of vaults

pier were treated as input, as was the horizontal stabilising force. Once the thrust pattern was found, the stress regime in each block could be examined separately.

Tunnels

There are many miles of brick lines tunnels, built in cut and cover, around London. They are frequently shallow enough to experience traffic loads and many of them also support buildings. One such passes just to the west of St Pancras station under both the railway lines and the Regent's canal. As part of the development of the Channel Tunnel rail link, a new small tunnel was to the bored under the existing one. The tunnel crosses at a shallow angle so complex settlement can be expected, first under one sidewall, then under the invert, then under the other sidewall. At the point where all this happens, a retaining wall stands on one sidewall of the tunnel. Above this wall run the lines of the Midland railway and the Channel Tunnel. At the lower level is a canal basin. The floor of the basin is roughly level with the crown of the tunnel.

A great deal of money was spent on the computer analysis of this tunnel. The result was a prediction of overstress even in the initial State. On the basis of this analysis, it was proposed to do over £5 million worth of work to support the old tunnel during construction of the new. Simple prediction of ground movement

suggested that it would not exceed 50 millimetres. This seemed unlikely to cause serious distress so a simple analysis was set up to explore the possible variations in soil pressure and the likely response of the tunnel.

Figure 4 shows the cross section of the tunnel with a predicted zone of thrust. This model illustrates the extent of interaction, which can be used. The user may choose four hinge points on the circumference of the tunnel, and then allow the tunnel to expand across one diameter. An extreme example of this displacement is shown in figure 5. This confirms that stress concentration can be expected in the tight radius corners at the bottom of the side walls. The visualisation also shows, however, that the high stresses only exist over a very short length, which means that modest plastic deformation of the mortar locally will accommodate and redistribute the stress.

Much of the interaction in this model is concerned with the transfer of load and with displacement. The tunnel is buoyant so to achieve vertical equilibrium it

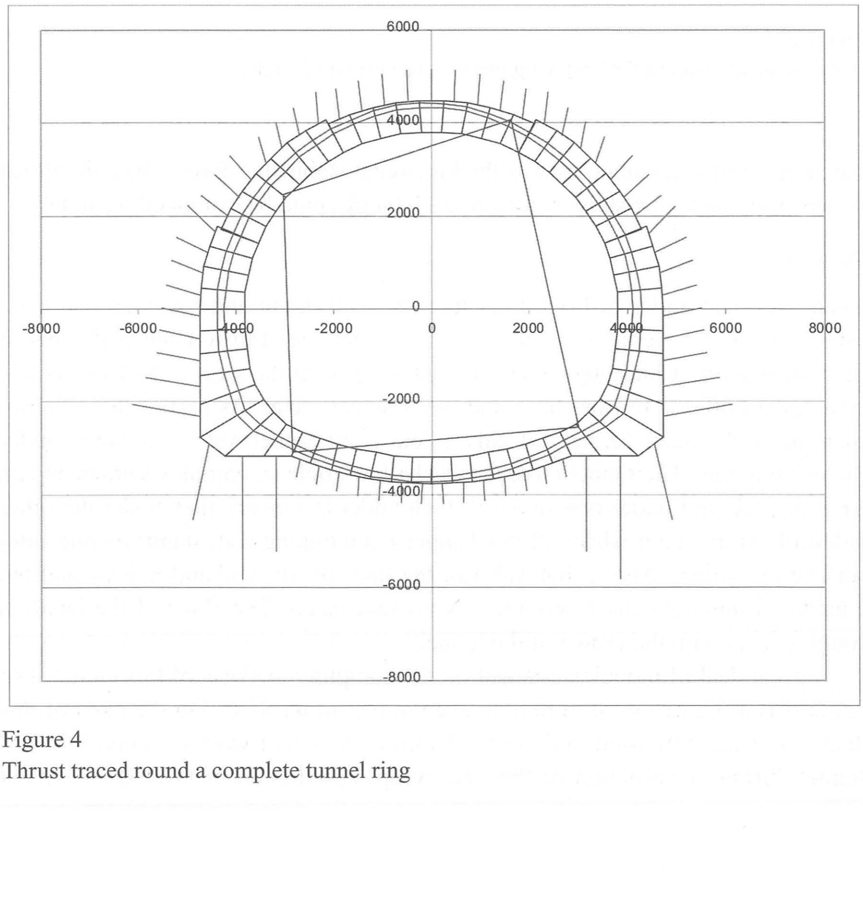

Figure 4
Thrust traced round a complete tunnel ring

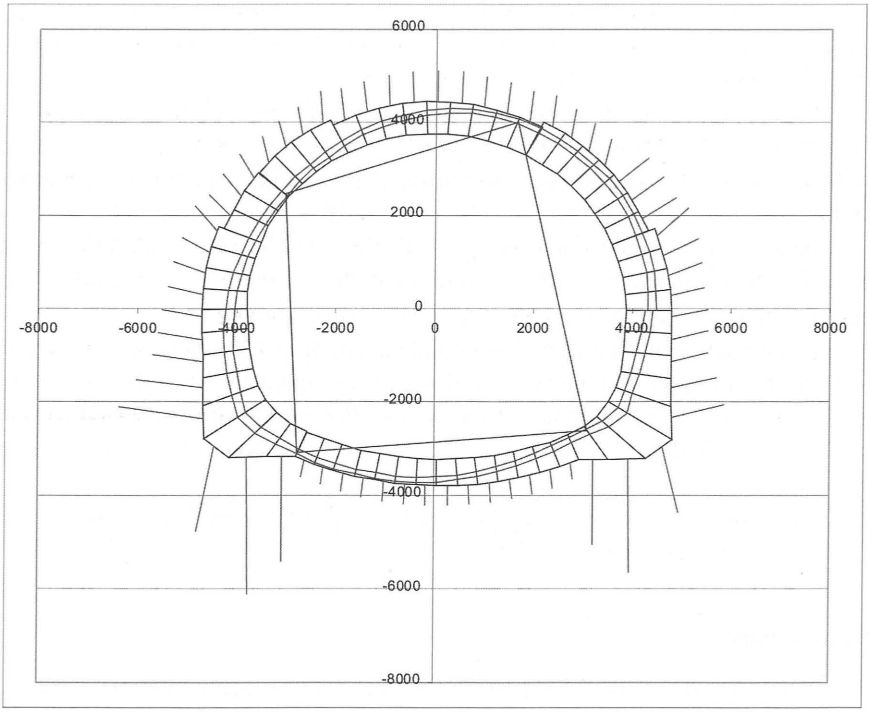

Figure 5
The tunnel with substantial settlement to the right. The quadrilateral marks the hinge positions.

is necessary to reduce the pressure on the invert, increase the pressure on the crown, or produce vertical friction on the side walls. Since many tunnels built at this time have no invert and this invert is both thin and flat. It seemed reasonable to assume that the vertical loads would concentrate on the wall foundations and the pressure on the invert will be relieved. An option to transfer load from the invert to the wall foundations was therefore included. When the new tunnel is under one wall of the old a vertical settlement of about 50 millimetres was anticipated. It is therefore reasonable to assume that the pressure under one wall would drop until the pressure under the other became high enough to cause settlement. Again, this transfer is achieved interactively by use of a scrollbar.

The transfer of load produces a torque on the tunnel. The manner of resistance to this torque is a matter of conjecture. It may be transferred on the tunnel by shear resistance. It may be resisted by skin friction. It might be accommodated by

redistribution of soil pressures. Most likely there will be a combination of all three mechanisms. Conveniently, it is not necessary to determine which carries the load, only to show that a support mechanism is available.

Surprisingly, a more difficult problem is the additional horizontal compression required to carry the zone of thrust through the hinges. In order to have confidence in this process, it was necessary to realise that the tunnel dilates as it cracks. With the two sides pushed apart by movement of the crown and invert additional horizontal force becomes immediately available. What is more, within the limits of plastic behaviour of the soil, the additional stress can be skewed upwards on one side and down on the other thus producing a restoring torque.

The interaction in this model is complex. The user can adjust all the forces and movement described above, and also vary the input thrust at the right hand side of the arch. With 10 points of interaction, the control is complex. The brain is, though, well able to cope with this complexity provided the visualisation is good and the response fast enough.

Full details of his work will be presented in a paper for the Institution of Civil Engineers in Britain in the near future.

Conclusions

This paper has shown how the power of simple analysis can be greatly extended using simple computer programs. The engineer is forced to recognise the difference between understanding, knowledge and confidence. Reliance on the plastic theorems is explicit and recognised rather than implicit.

Jacques Heyman had been working on arching structures for many years by the time the author became interested in the subject. Much of the thinking presented here is built directly on the ideas presented by Heyman in his various papers and books.

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Thomas Young's theory of the arch. His analysis of Telford's design for an iron arch of 600 feet over the Thames in London

Santiago Huerta

The main lines of the development of arch theory are well known. The works of Poncelet (1852) and Winkler (1879) give a good review of the early theories from the XVIIth to the mid-XIXth century. Those theories refer to masonry arches (often called "rigid"). The theory of the "elastic" arch developed during the XIXth century and was applied first to iron and wooden arches; after the 1880's it was applied to any kind of arches. A detailed study of the history of the elastic theory may be found in Mairle (1933) and a good review of the fundamental lines in Hertwig (1941), Timoshenko (1953) and Charlton (1982). Heyman (1972, 1998) has studied the evolution of arch theory within the frame of limit analysis, and has placed it rigorously within the general frame of the modern theory of structures. A recent article by Kurrer (1997) covers both the history of rigid (masonry) and elastic theories. Finally, Foce (Becchi and Foce 2002) has contributed a new historical review and, more important, has compiled a comprehensive bibliography of the primary sources.

However, if the overall picture is clear, some details should still be investigated. Little parts of the canvas are still blurred and certain contributions, steps on the ladder of progress, have been forgotten. This is the case with the contribution of Thomas Young (1773–1829) to arch theory, which is not even mentioned in any of the works cited above. The omission is amply justified by Young's obscure prose and his eccentric way of publishing. His work, though considered important by some eminent contemporary engineers like Rennie, was not understood and rapidly forgotten. Young's arch theory exerted apparently no influence. But it is a fact that he had a deep understanding of arch behaviour (his theory was basi-

cally correct) and was well ahead from his contemporaries. The culmination of his work on arches is the article *Bridge* for the *Supplement* to the fourth edition of the Encyclopaedia Britannica, published in 1817. In it he exposed first the theory and then, as a tour de force, applied it to the analysis of Telford's unbuilt design for a great iron arch of 600 feet over the Thames (1800).

To put in context the work of Young a few words should be said about the state of the art of arch theory ca. 1800. Telford's design episode will be also revised because it served as a "touchstone" for the state of this theory in Britain and, also, because it could have triggered Young's interest in arch bridge design.

Arch theory circa 1800

At the beginning of the XIXth century there were two approaches to arch analysis: 1) the "equilibration theory", and, what we may call, 2) the "point of rupture" theory. The first originated and developed in Great Britain and the second in France. Both theories were considered essentially as different approaches until the 1840's when, thanks to the correct definition of the concept of "line of thrust" it was understood that both theories were equivalent.

Equilibration theory

The equilibration theory originated in Hooke's analogy (1675) between hanging chains and arches: "As hangs the flexible cable, so but inverted will stand the rigid arch". The statics of cables and arches is essentially the same, and the form of the catenary is the ideal form for an arch of uniform thickness. The architect or engineer following Hooke's approach would like to make the arch of the same form of the corresponding hanging chain. The matter was tackled mathematically by many English mathematicians and engineers during the XVIIIth century, and applied to arch analysis, for example, by Emerson (1754) and Hutton (1772, 1812). There were two basic problems: 1) to find the intrados for a given extrados; 2) to find the extrados for a given intrados, figure 1 (a) and (b).

In the case of a bridge, the load on the chain (the arch ring) was the weight of the arch plus the load of the filling and road. Being the last sensibly horizontal, the form of the arch should be such that the load in every point is proportional to the vertical distance of this point to an horizontal line of extrados. In 1801 Robison proposed a hanging model, figure 1 (c), with rods representing the load, which expressed clearly the philosophy of bridge design following the equilibration theory. The physical interpretation of the equilibration arch is a series of smooth voussoirs with the joints always normal to the curve of intrados. Both approaches lead to the same result: a certain fixed form (the intrados) for the transmission of the thrusts, the curve of equilibrium. The theory gives no information

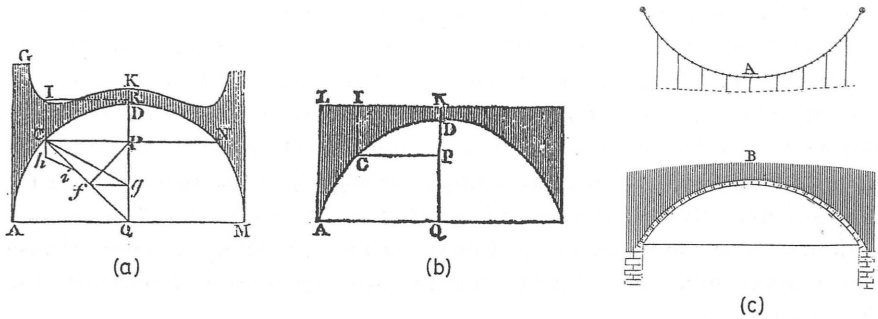


Figure 1
The two main problems of the equilibration theory: (a) To find the curve of extrados for a given intrados; and (b) to find the form of a intrados for a given extrados. (Hutton 1812). In figure (c) the model suggested by Robison in 1801 to solve the second case (Young 1807)

about the thickness of the arch and does not explain common phenomena as the craking of arches. Of course, the equilibration theory permits to calculate the thrust of the arch, which is the reaction at the end of the inverted chain, known in position, magnitude and direction. However most English contributions did not tackle the problem of buttress.

The problem is that any change of the load will distort the curve of equilibrium, which will fit no longer with the built arch (and Robison's model may be used to check this assertion experimentally). The case is specially serious in bridge design, a bridge being precisely an structure for the passing of moving loads. Besides, the curves obtained were difficult to construct with simple geometrical methods, and therefore not adequate for the common practice of building. However, these inconvenients did not deter engineers and mathematicians who continued to expend a lot of labour and ingenuity in studying every conceivable situation for arches, first, and then for domes and vaults.

"Point of rupture" theory

The second theory originated in France and La Hire (1712) made the first contribution. The approach is not directed to the study of the form of the arch but to obtain its thrust in order to calculate the depth of the abutments. La Hire observes that in a collapsed arch or barrel vault the inferior part remains united to the abutment, marking the "point of rupture" of the arch or barrel vault. The thrust must pass through this point and be tangent to the intrados and, once locat-

In the 1790's the growth of trade made necessary a large reform of the Port of London. In the years 1798 and 1799 several proposals were made to replace the old London Bridge in order to admit the passage of cargo ships. All the proposals were influenced by the success of the iron bridge built at Sunderland in 1796 with a span of 236 feet (72 m); this was a proof of the feasibility of using cast iron in the building of large bridge arches. The Select Committee formed to study the reform advertised for designs for a new bridge with 65 feet height above high water, suitable for passage of 200 ton ships. Several designs were presented: Thomas Wilson (an iron bridge of three arches, 220, 240 and 220 feet span), Ralph Dodd (a monumental masonry bridge) and Telford and Douglass (three designs of three and five iron arches). George Dance proposed a drawbridge. All the projects were prepared for publication in the Third Report of the Select Committee of 28th July 1800.

Then a report by William Jessop attracted the attention of the Committee. Jessop argued that to allow the passage of cargo ships the river should be dredged from the actual deep of 6 to 10 feet to 13 feet in the middle and that to maintain the velocity of water the river should be narrowed to 600 feet, constructing embankments and warves (Ruddock 1979, 156). The reduction to the span to 600 feet prompted Telford and Douglass (though the design of the bridge must be attributed to Telford[1]) to present in the autumn of 1800 a new project with a single cast-iron arch covering the whole span. The design arrived too late to be included in the Third Report, but a plate with the plan and elevation (Fig. 2) and a report and estimates were issued in a Supplemental Appendix. A model of the bridge was also made. The Committee expressed his admiration for the new design:

> The obvious advantages which would be obtained if the Communications could be effected by Means of Single Arch, as well as the Magnificence of the proposed Structure, appeared to give the . . . Design a particular Claim to the Notice of Your Committee; yet the Attempt was of so novel a Nature, that they thought it absolutely necessary for their own Information, as well as for the Purpose of affording some Grounds upon which the House might hereafter form their Judgement as to its Expediency, to request the Opinions of some of the Persons most eminent in Great Britain for their theoretic as well as Practical Knowledge of such Subjects. (Fourth Report 1801)

The Committee draw up twenty-one questions to be sent with the design and two additional explanatory drawings of the framing of the ironwork (Fig. 3) to a list of eminent experts.[2] The experts selected included three groups of persons: scientists and mathematicians, eminent engineers and iron makers. The strategy of the Committee was to seek the correct answers combining the judgements of

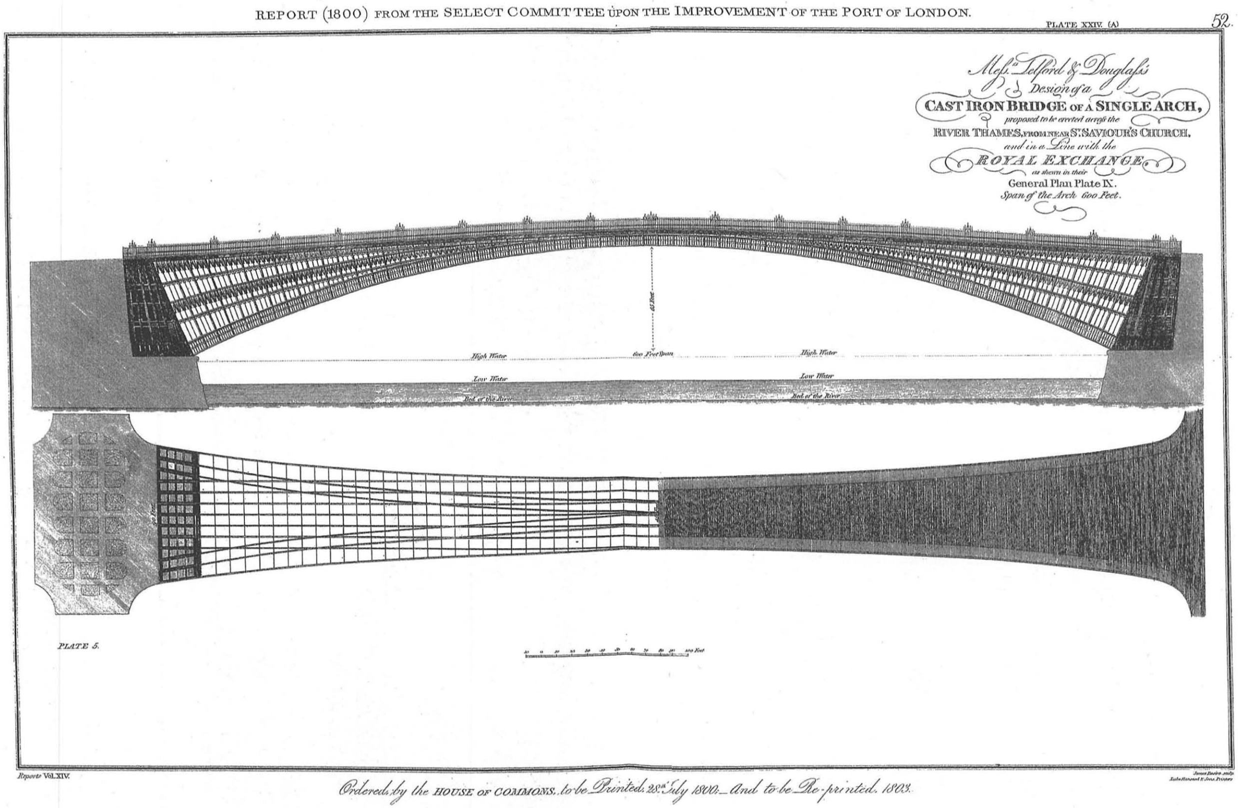


Figure 2
Plan and elevation of Telford's design of 1800 for an iron arch bridge of 600 feet to be built over the Thames in London. (Third Report 1800)

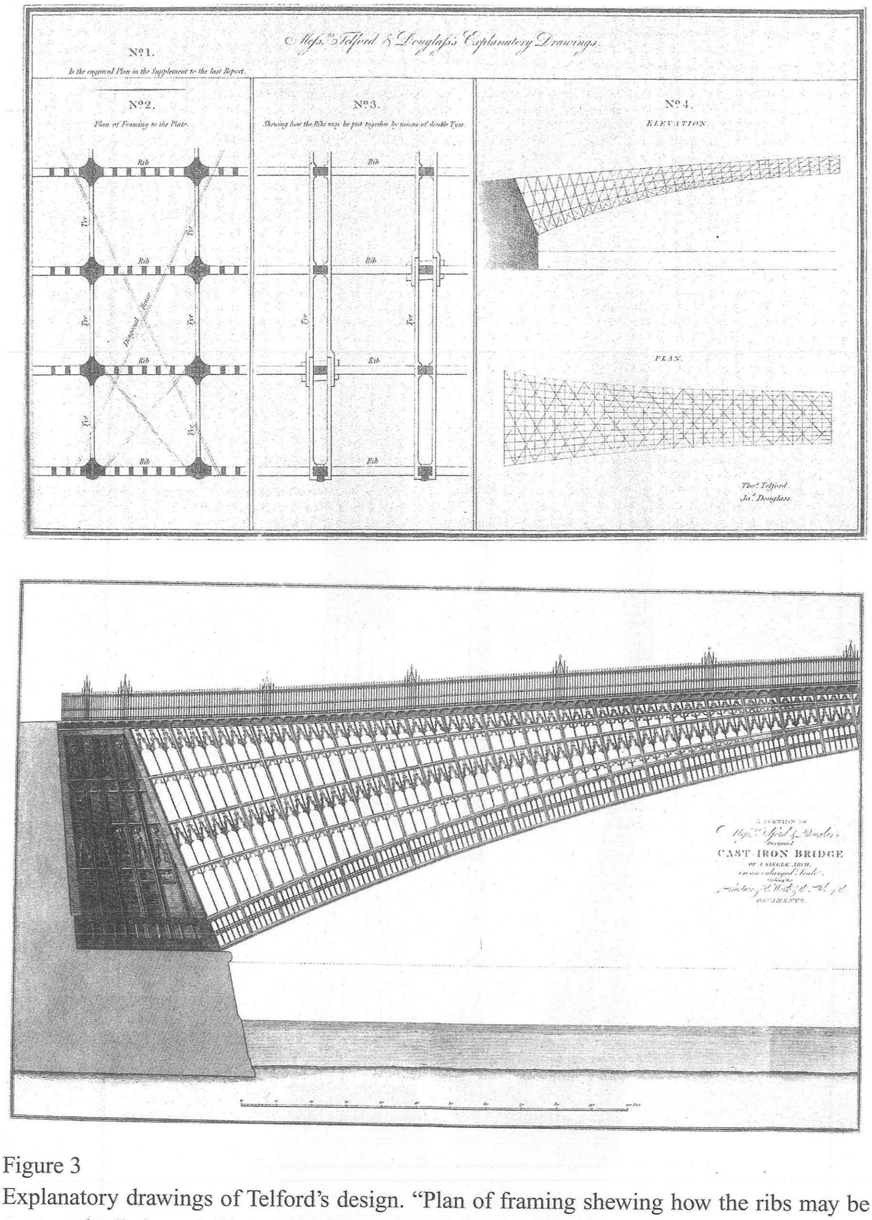


Figure 3
Explanatory drawings of Telford's design. "Plan of framing shewing how the ribs may be put together". (Fourth Report 1801)

all these approaches. In fact, it was Telford himself who drafted the questions while corresponding with several of the selected experts. It is obvious that he dedicated a lot of attention to the matter (he made four drafts) and the list constitutes an exhaustive questionnaire in which all the matters relating to the design of a bridge are considered (the list is reproduced in an Appendix at the end of this paper). The questions were sent early in April 1801 and nearly all replies were dated at the end of this month. The questions and answers, together with the new drawings, were issued in the Fourth Report on 3rd June 1801.

The answers received must have supposed a great deception both to the Committee and to Telford himself. There is no space here to enter in detail in the matter (for a discussion see Dorn 1970; brief comments in Skempton 1980) but it was evident that the state of knowledge of structural theory was insufficient to answer the precise and intelligent questions posed by Telford. Quoting Peacock (1855, 422): "The answers which were given were singularly humiliating to the pride of philosophy: they were not only altogether at variance with each other, but in very instance incomplete and unsatisfactory".

Telford pressed forward in favour of his design and in the summer of 1801 an splendid engraving with a large view of the design was published, which attracted a lot of attention from the public. The same year, he published, also, an article in the prestigious *Philosophical Magazine*. Still a year later Telford apparently have received notes of congratulation from the King (Ruddock 1979), but the proposal was finally abandoned and eventually the new London Bridge was built as a traditional masonry bridge of three arches. This must have been an enormous deception to Telford and maybe a sign of this is that no mention is made of this episode in his autobiography (Rickman 1838).

The reasons for the abandon of such a magnificent design were not made explicit. It is a fact that most of the experts have a favourable opinion as to feasibility of the design, Question XX of the list, though they were unable to justify it. Both Ruddock and Skempton believe that the main reason would have been the cost and complexity of building the long approaches to the bridge. However Dorn (1970), though considering also the economical aspect, says that "a suspicion lingers that the project was undermined by the inability of the Committee's respondents to provide any convincing assurance that practice harmonised with theory in Telford's majestic design". Indeed, the questions were so clear and straightforward that the inability to be answered would have caused suspicion in any cultivated man.

The whole episode provoked an awakening in the interest in arch theory in Britain. Some of the respondents published articles and books on the subject. Hutton urged to make a reprint on 1801 of his treatise on bridges of 1772 and in his *Tracts* of 1812 included a new improved and revised edition of it. Southern

(1801) published a paper on the equilibrium of arches. The same year Atwood published *A Dissertation on the construction and properties of arches*, which was followed in 1804 by a S*upplement*. In 1811 an anonymous correspondent published in the *Philosophical Magazine* a paper with the expressive title "Some Account of the different Theories of Arches or Vaults, and of Domes, and of the Authors who have written on this most delicate and important Application of Mathematical Science" (Some Account 1811). In this paper, maybe for the first time in England, a detailed account of the French theories of arches is given. However, none of this contributions supposed a remarkable advance on the state of the theory which would have permitted to answer the 21 questions posed by Telford on bridge analysis and design.

Thomas Young's theory of the arch

Against this background should we see Thomas Young's contribution to arch theory. He was interested in arch theory for a period of fifteen years, between 1801 when he accepted to deliver the Lectures for the Royal Institution (this marks the beginning of his interest on the Mechanical Arts), and 1816 when he finished writing his article "Bridge" published the next year in the *Supplement to the fourth edition of the Encyclopaedia Britannica*. An study of the evolution of Young's studies on arches, though concentrated only in two publications, the *Lectures* of 1807 and an obscure paper "On the structure of covered ways", published anonymously in 1807, will require more space than is allowed in the present book. Therefore we will concentrate in the article "Bridge" which contains his whole theory on arches.[3]

The article was included by Peacock (1855) in his edition of the *Miscellaneous works of the late Thomas Young*, but with some important modifications. First, he reduced considerably the number of figures; only the first 7 figures of the first of the three plates of the original article were included, which form the upper part of the first original Plate reproduced in figure 4. Secondly, he eliminated the comments of the figures included. And, finally, the sixth and last section of the article was completely suppressed. Particularly, this last suppression makes difficult to understand some of Young's propositions which were applied in this section to the analysis of the bridges of Southwark and Waterloo. However, as it is much easier to consult Peacock's edition (which, besides, has been reprinted in 2003) than the original article in the Encyclopaedia, in what follows all the references within brackets are to the pages in the *Miscellaneous works*, except when otherwise specified.

Young is very explicit about his intentions, and the article begins:

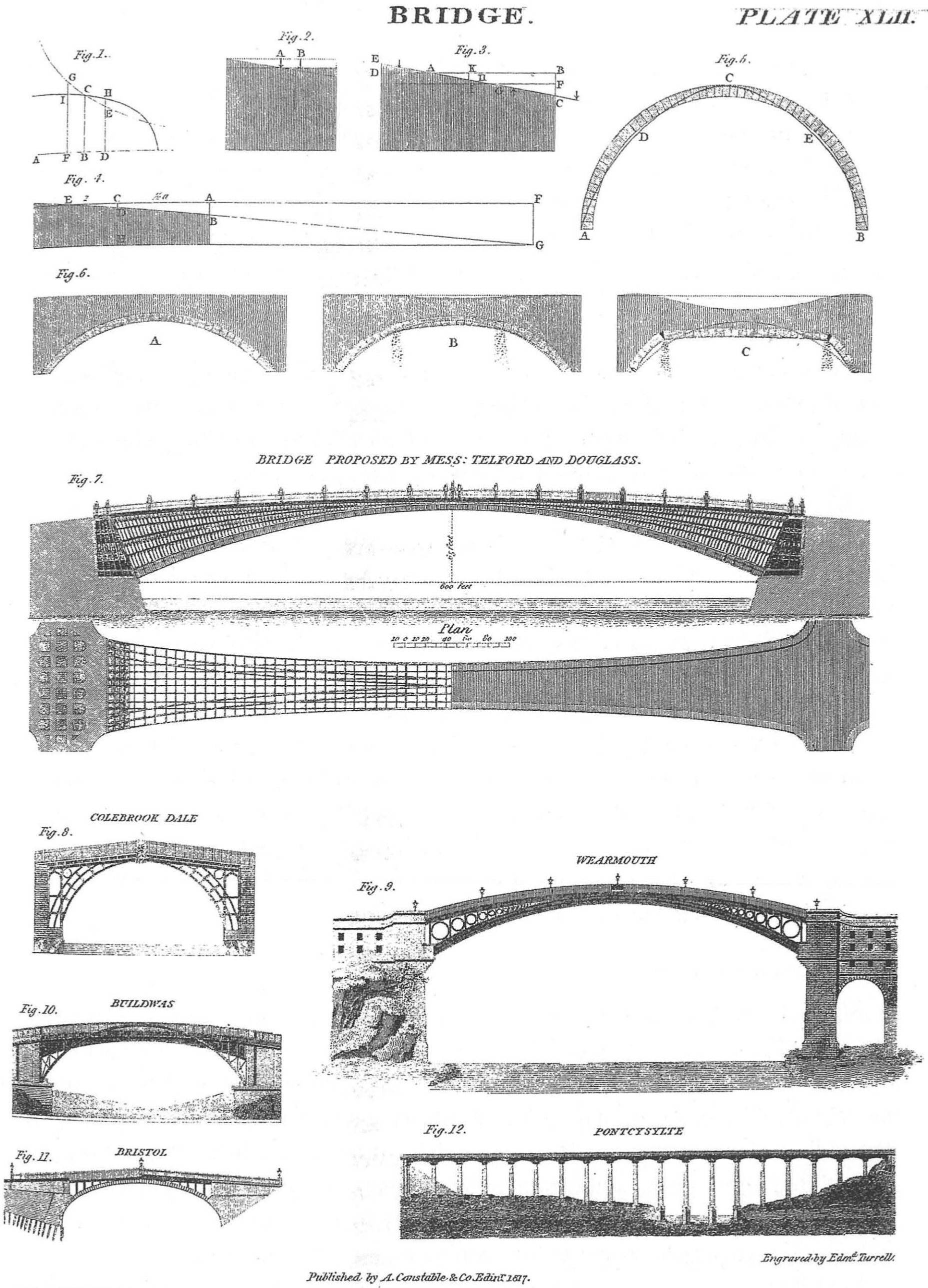


Figure 4
First Plate of the article "Bridge" written by Thomas Young for the *Supplement* to the 4th Edition of the Encyclopaedia Britannica. (Young 1824 [1817])

> The mathematical theory of the structure of bridges has been a favourite subject with mechanical philosophers; it gives scope to some of the most refined and elegant applications of science to practical utility; and at the same time that its progressive improvement exhibits an example of the very slow steps by which speculation has sometimes followed execution, it enables us to look forwards with perfect confidence to that more desirable state of human knowledge, in which the calculations of the mathematician are authorised to direct the operations of the artificer with security, instead of watching with servility the progress of his labours. (194)

The criticism to the actual situation of impotence of the theory to explain the normal practical procedures or to check the feasibility of new designs is clearly stated, and so it is the ambitious objective of formulating a theory which could put an end to this state of affairs, harmonising theory and practice.

The article is divided in six parts. The first three contains the theory of arches: 1) “Resistance of materials”, 2) “The equilibrium of arches” and 3) “The effects of friction”. The fourth part contains some “Earlier historical details” (a discussion on the origin of the arch and a review of “the most important operations” in bridge building, extracted from Smeaton Reports). The fifth part contains “An account of the discussions which have taken place respecting the improvement of the port of London”. In fact, this part is dedicated to answer in detail to the 21 questions of the Select Committee, applying the theory previously exposed in the first three parts. Finally the sixth part is “A description of some of the most remarkable bridges which have been erected in modern times”. In this part, after a brief history of the iron bridges, the theory of arches is applied to analyze in detail the bridges of Southwark and Waterloo. In all, of the 23 pages of the article, 19 pages are dedicated to strictly structural matters.

Resistance of materials

In this part Young particularize his theory of “passive strength” already expounded in his *Lectures* of 1807 with a view to its application to arches. Young makes an effort to explain the theory in rigorous terms. The method used by Young is the “classical” method of stating a proposition (named alphabetically from A to Z) and then demonstrating it. This way of exposition makes difficult to follow the general line of reasoning and results particularly exasperating to a modern reader. The propositions though formulated in a general manner are directed to study the arch problem: a curved structure functioning mainly in compression.

First he states the proportionality between tensions and deformations and to justify this he expounds a theory of cohesive and repulsive molecular forces and states that even if the law of this forces is not linear (*fig.* 1 in figure 5), the effect will be proportional for a small “change of dimensions”. (196)

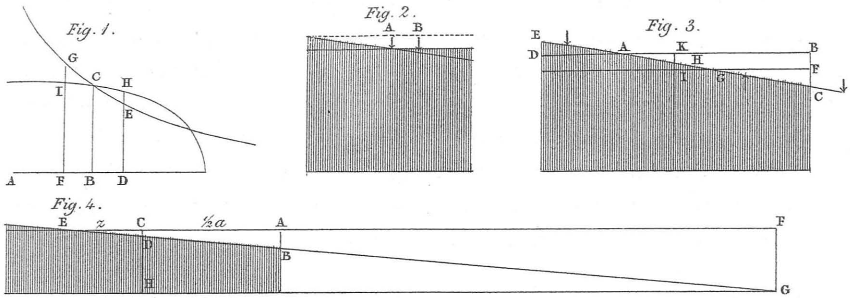


Figure 5
Drawings on "Resistance of materials". Young is concerned with the "compression" or "extension" of a joint which remains plane after deformation. He treats deformations not stresses. (Detail of Figure 4 above)

He then treats the eccentric compression of a block and states and begins considering the limit position of an eccentric force so that all the section remains in compression and the corresponding increase in the stresses. However, the way he expressed the problem is as follows: "The strength of block or beam must be reduced to one half, before its cohesive and repulsive forces can both be called into action". A modern engineer may have no difficulty in interpreting this: Young is obviously referring to the "middle third" concept and the maximum stress is double as the mean stress. To demonstrate this, Young assumes explicitly that plane sections remain plane after the deformation. It follows that the deformations (compressions or extensions) varies linearly and "consequently the forces may always be represented, like the pressure of a fluid, at different depths, by the ordinates of a triangle; and their result may be considered as concentrated in the centre of gravity of the triangle, or of such of its portions as are contained within the depth of the substance." (197) Here Young is struggling with the concept of stress and he uses the analogy of the pressure of a fluid. However he tries always to speak in terms of deformations, the "forces" or "pressures" being always proportional to them, as stated in the first proposition, and not of stresses (*fig.* 2 in figure 5).

The next proposition states that "the compression or the extension of the axis of the block or beam is always proportional to the force, reduced to the direction of the axis, at whatever distance it may be applied". (198) The deformation of the axis is always equal to the mean deformation, produced by the normal component of the force applied in the middle of the section. The transverse component of the force will be resisted by "lateral adhesion" (shear) and if the force is normal to the axis "the length of the axis will remain unaltered".

Then Young proceeds to locate the neutral point for this general force placed at any distance:"The distance of the neutral point from the axis is to the depth, as the depth to twelve times the distance of the force, measured in the transverse section". In an algebraical form

$$z = \frac{a^2}{12y} \tag{1}$$

where z is the distance of the neutral point from the axis, a is the depth of the section and y is the distance of the point of application of the force to the axis. Young's demonstration is based in the proportionality of the stress resultants and the triangular form of the stress blocks; it is not easy to follow even knowing that it is correct.

The next proposition tries to relate the increase of the normal stresses in terms of the distance of the force from the axis: "The power of a given force to crush a block, is increased by its removal from the axis, supposing its direction unaltered, in the same proportion as the depth of the block is increased by the addition of six times the distance of the point of application of the force, measured in the

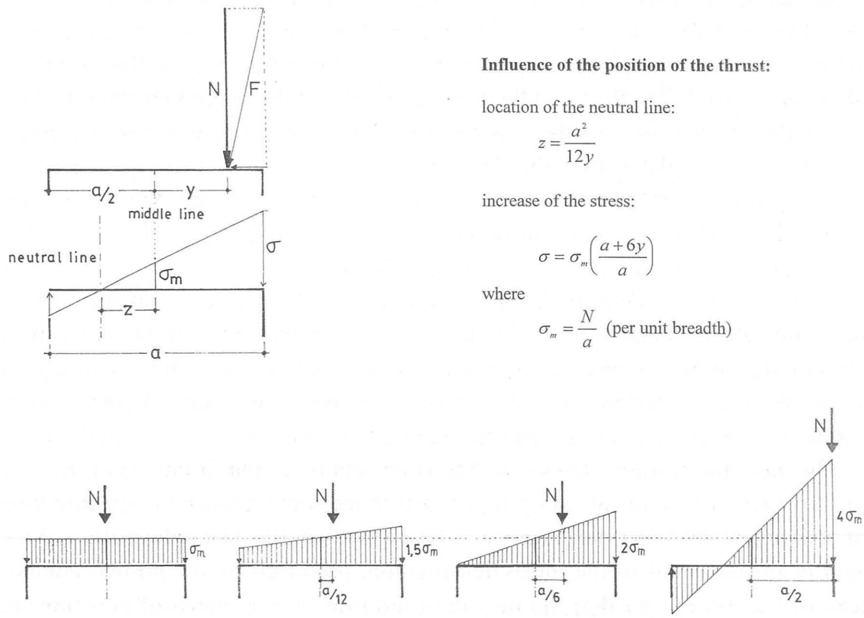


Figure 6
Young's propositions on "Resistance of materials" expressed in modern terms (stresses).

transverse section". (199) Young is referring to the increase of the stresses due to the eccentrity of the load. In modern terms, if we call σ_m the mean compressive stress produced by the force applied in the center of the section, the removal of the force at a distance y will produce a stress σ given by

$$\sigma = \sigma_m \left(\frac{a + 6y}{a} \right) \qquad (2)$$

Young demonstrates the assertion, again, for similar triangles (*fig.* 3 in figure 5). Therefore, now we are in the situation to ascertain the "strength" (stress distribution) of any section acted by any force located at any distance, figure 6.

On the equilibrium of arches

The next Section of the article treats the equilibrium of arches, i.e. it is an study of the definition and mathematical properties of the curves of equilibrium (lines of thrust), but with a view to their application in the analysis and design of actual bridges.

DEFINITION OF LINE OF THRUST: To study the equilibrium of arches Young "proceed to inquire into the mode of determining the situation and properties of the curve of equilibrium, which represents, for every part of a system of bodies supporting each other, the general direction of their mutual pressure". (204) Here is, twenty years before the official date of 1835, the definition of line of thrust. Young has liberated himself from the straitjacket of the equilibration theory (vertical loads, thrust following the line of intrados) and speaks freely of the equilibrium of a system of bodies in contact.[4]

Young is well aware that the form of the curve of equilibrium (in what follows we will use this term) depends on the family of planes of joint considered: ". . . it is obvious that the forces . . . may vary very sensibly in their proportion if we consider the joint operation on a vertical or on a oblique plane". (205) However, he immediately remarks that " . . . if the depth of the substance be inconsiderable, this difference will be wholly imperceptible, and in practice it may generally be neglected without inconvenience; calculating the curve upon the supposition of a series of joints in a vertical direction". (205)

He explains, however, the method to study any particular joint: "if we wish to be very accurate, we must attend to the actual direction of the joints in the determination of the curve, and must consider, in the case of a bridge, the whole weight of the structure terminated by a given arch stone, with the materials which it supports, as determining the direction of the curve of equilibrium where it meets the given joint . . . this consideration being as necessary for *determining the circumstances under which the joints will open, as for the more*

imaginary possibibility of the stones sliding upwards or downwards". (italics are mine)

Young, then, has a perfect grasp of the concept of line of thrust but, instead of losing himself in the mathematical intricacies of the problem (considering different families of planes of joint); he considers a good approximation the assumption of vertical joints, but bearing in mind the possibility of studying in more detail any particular joint.

Then Young formulates a series of propositions addressed to apply his ideas of curves of equilibrium to different types of loads.

CURVE OF EQUILIBRIUM IN A FLAT ARCH: He begins with the straight arch, the platebande, and deduces that the form of the curve of equilibrium must be parabolic. Young uses this simple example of the platebande to make clear his ideas of the curve of equilibrium. He remarks that the thrust in the central joint must be horizontal and, then, chooses a system of vertical joints "which is the only way in which we can easily obtain a regular result". (206) For a block cut at distance x from the middle, calling the ordinates y, as the weight is proportional to x, it is evident that $x = m\dfrac{dy}{dx}$, and integrating, $({}^{1}/_{2})x^2 = my$, which is the equation of a parabola. Now Young alludes to the conventional representation of this line as an inverted funicular polygon (as we call it nowadays): "It is usual in such cases to consider the thrusts rectilinear throughout, and as meeting in the vertical line passing through the centre of gravity of each block; but this mode of representation is evidently only a convenient compendium".

GENERAL EQUATION OF THE CURVE OF EQUILIBRIUM: In the next proposition Young gives the general equation of the curve of equilibrium for any symmetrical vertical distribution of the load, considering vertical joints: "In every structure supported by abutments, the tangent of the inclination of the curve of equilibrium to the horizon is proportional to the weight of the parts interposed between the given point and the middle of the structure". (207) He notices that in bridges the loads may not act entirely in a vertical way, some materials exerting a lateral pressure also; due to the symmetry, this does not affect the general truth of the assertion, though the form of the curve of equilibrium will vary slightly. He discourages the use of such materials for the filling.

Then, we have:

$$\int wdx = mt = \frac{dy}{dx} \tag{3}$$

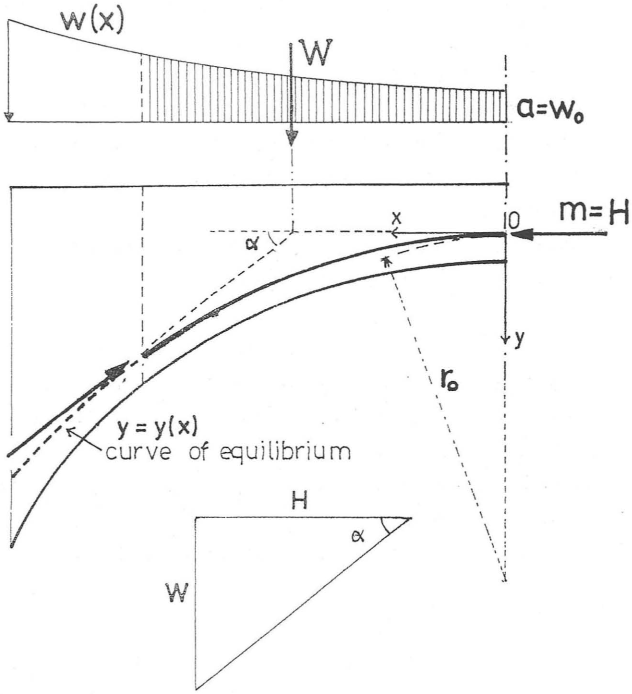


Figura 7
Line of thrust or curve of equilibrium for a symmetrical arch, which supports a vertical load, considering vertical joints.

where "w is the height of uniform matter, pressing on the arch at the horizontal distance x from the vertex, t is the tangent of the inclination of the curve of equilibrium (tan), y is the vertical ordinate, and m is a quantity proportional to the lateral thrust, or horizontal thrust". If we consider a vertical load, m is equal to the horizonal thrust, figure 7.

Young now studies the properties of curvature of the curve of equilibrium in relation with the load and the inclination of the thrust and extracts two corollaries relating to circular and parabolic curves of equilibrium: "The radius of curvature of the curve of equilibrium is inversely as the load on each part, and directly as the cube of the secant of the angle of inclination to the horizon". (208)

The general expression of the radius of curvature is,

$$r = \frac{(dz)^3}{dxd^2 y}$$

where dz is a differential element of the curve (following Young's notation). But $mdy = \int wdx$ and it follows $m\, d^2y = w(dx)^2$; $dz = dx\sqrt{(1 + t^2)}$, and subsituting in the above equation of the radius,

$$r = \frac{m}{w}(1 + t^2)^{\frac{3}{2}} = \frac{m}{w}(1 + (\tan\alpha)^2)^{\frac{3}{2}} = \frac{m}{w}(\sec\alpha)^3 \tag{4}$$

and at the crown, $\sec\alpha = 1$, $w = w_o$

$$r_0 = \frac{m}{w_0}. \tag{5}$$

HORIZONTAL EXTRADOS AND INTRADOS TERMINATED WITH THE CURVE OF EQUILIBRIUM: This was the usual assumption for bridges in many previous arch treatises. He expresses the result in the form of a proposition: "For a horizontal extrados, and an intrados terminated by the curve itself, which, however, is a supposition merely theoretical, the equation of the curve is

$$x = \sqrt{m}\ln\left(\frac{y + \sqrt{y^2 - a^2}}{a}\right)." \tag{6}$$

In this case the load $w = y$ and, for a depth of the arch a at the keystone, Young obtains the equation of the abscises in function of the ordinates because the integration is much more easy. The result is correct and obviously to obtain the different points of the curve for different ordinates only a table of neperian logarithms is needed. However, Young makes clear that "such a calculation is by no means so immediately applicable to practice, as has generally been sup-

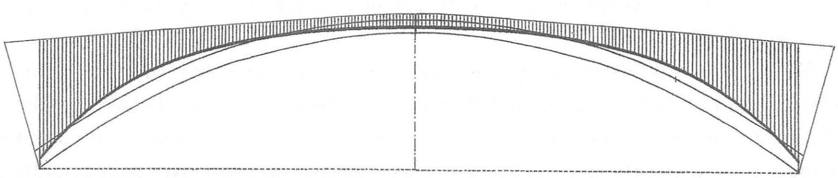


Figure 8
Curve of equilibrium for a load which is proportional to the vertical distance to a horizontal extrados. In a "typical" arch, maybe of circular form, if the curve pass through the middle of the joints at the keystone and abutments (Young's usual assumption), the curve lies completely outside of ring of normal thickness.

posed; for the curve of equilibrium will always be so distant from the intrados at the abutments, as to derange the whole distribution of the forces concerned". (209) In fact, this approach contradicts the main objective of Young, which is to free the curve of equilibrium from the "straitjacket" of the intrados. Besides, if we obtain this curve, passing through the middle of the joints at the crown and the springings (Young's usual assumption), the form of the curve of equilibrium differs so much from that of the arch as to be completely useless, figure 8.

PARABOLIC LOAD: This is the most important proposition of this part. Young realizes that to handle in a convenient way the different curves for a given load, this load should have a mathematical definition which leads to a simple integration. Also, it should be sufficienlty flexible to adapt to the real loads in a bridge and to take into account the inclination of the joints, if considered necessary. He decides that a parabolic load fulfils both conditions and gives the corresponding equation : "If the load on each point of an arch be expressed by the equation $w = a + bx\approx$, the equation for the curve of equilibrium will be

$$my = \frac{1}{2}ax^2 + \frac{1}{12}bx^4\text{''}. \tag{7}$$

The whole load $W = \int wdx = ax + (1/3)bx^3$. Now, $m\ (dy/dx) = ax + (1/3)bx^3$, and integrating the above cited expression is obtained. Young cites explicitly its advantages: "This expression will, in general, be found sufficiently accurate for calculating the form of the curve of equilibrium in practical cases; and it may easily be made to comprehend the increase of the load from the obliquity of the arch-stones". (210)

Given the ordinate y at the abutments (that is the height of the curve of equilibrium between its springings and the point of horizontal tangent at the joint of the keystone) it is easy to obtain the value of the horizontal thrust m. And at the keystone $w = a$ and the radius of curvature is $r = m/a$ as the secant of zero is the unity.

LOAD TERMINATED BY A CIRCULAR OR ELLIPTICAL ARC: He gives the equation of the curve of equilibrium for a load defined by a horizontal extrados terminated by a circular or elliptical intrados. (211) This is the case of a masonry bridge when the filling has the same specific weight as the arch-stones, which will be in a real bridge only a very crude approximation. "When the load is terminated by a circular or elliptical arc, $w = a + nb - n\sqrt{b^2 - x^2}$ and

$$my = \frac{1}{2}(a + nb)\,x^2 - \frac{1}{2}nb^2\,x\arcsin\left(\frac{x}{b}\right) - \\ -\frac{1}{2}nb^2\sqrt{b^2 - x^2} + \frac{1}{6}n\,(b^2 - x^2)^{\frac{3}{2}} + \frac{1}{3}nb^3\text{''}. \tag{8}$$

The expression of the load may be immediately deduced for a circular form and the coefficient n represents only an stretching to obtain an ellipse (for the circular form $n = 1$). Young makes correctly the corresponding integrals, obtaining the above cited mathematical expression. The radius of curvature at the vertex will be again $r = m/a$. Young will apply later this expression in the calculation of the curve of equilibrium of Blackfriars Bridge.

DISCUSSION ON CURVES OF EQUILIBRIUM WITHOUT FRICTION: Young states that the condition for the equilibrium of an arch without friction is that "a curve of equilibrium, perpendicular to all the surfaces of the joints, must be capable of being drawn within the substance of the blocks". (212) This is, of course, the essence of the equilibration theory and Young dedicates two pages to criticize this, preparing the reader for his last proposition on the effects of friction. In fact, in this paragraph, he will discuss the formation of hinges, the corresponding diminution of "strength" (the increase of the stresses) and the way of collapse of masonry bridges.

He asserts that, in practice, the possibility of failure by sliding is almost impossible, but "if the curve [of equilibrium] . . . be directed to a point in its plane beyond the limits of the substance, the joint will open at its remoter end, unless it be secured by the cohesion of the cements, and the structure will either wholly fall, or continue to stand in a new form." (212) It appears that for the first time there is established a relationship between line of thrusts and the formation of hinges, and the possibility of an arch to adapt to the movements by cracking. Young does not expand the statement but refers the reader to the *fig.* 5 in figure 4. (In the *Miscellaneous papers* Peacock eliminated all the comments on the plates.) He says that, in this situation, "the joints in the neighbourhood of D [and E] will be incapable of resisting the pressure in the direction of the curve CD, and must tend to turn on their internal terminations as centres, and to open externally" (Young 1824, 520).

Then Young comments the reduction of strength when the curve of equilibrium touches the limit of the arch; in this situation the stress is four times higher as the mean stress, equation (2). But Young is well aware that this is in the hypothesis of plane deformation and the existence of cohesion (tensile strength) and that in

reality "the diminution of strength will probably be considerably greater than is here supposed, whenever the curve approaches to the intrados of the arch". (213)

Finally, he discussed the problem of the process of collapse of a real bridge, *fig. 6* in figure 4. He is not considering a rigid material and, therefore, the deformations are not concentrated exclusively at the hinges (though the cracks at the haunches are clearly drawn). In fact, he is trying to explain the results of some experiments reported by Robison (1801) with the help of his new ideas on curves of equilibrium.

EFFECT OF FRICTION: In this part Young resumes the main consequences of friction in respect to the stability of arches and of masonry structures in general: "The friction or adhesion of the substances, employed in Architecture, is of the most material consequence for insuring the stability of the works constructed with them". (214) With respect to arches, he realized crucial importance of friction to let the curve of equilibrium move within the arch. The corresponding (and last, of the theory of arches) proposition resumes the main aspects: "The joints of an arch, composed of materials subject to friction, may be situated in any direction lying within the limits of the angle of repose [friction] . . ." (215)

He concludes "that the direction of the joints can never determine the direction of the curve of equilibrium crossing them, since the friction will always enable them to transmit the thrust in a direction varying very considerably from the perpendicular", (216) though he adverts also, that sometimes the true direction of the joints should be taken into account, as they affect the form of the curve of equilibrium and the direction of the thrusts and in this case: " . . . with respect to any particular joint, of which we wish to ascertain the stability independent of the friction, it would be desirable to collect the result of the elements, of which that curve is the representative, with a proper regard to its direction."

Analysis of Telford's design for London Bridge

The objective of Young in writing the article Bridge was not to give another mathematical discussion on the theory of arches, similar to that of Hutton or Atwood. He wants to develop a theory to be applicable to the design of real bridges. Therefore, after the theoretical parts on Strength of Materials and Theory of Arches, he passed on to apply his theory to real cases. He first addresses his attention to Telford's design. His appreciation of the answers given by the experts to the questions posed by the Select Committee, which he no doubt read with great care, is unambiguous: " . . . the results of these inquiries are not a little humiliating to the admirers of abstract reasoning and of geometrical evidence; and

it would be difficult to find a greater discordance in the most heterodox professions of faith, or in the most capricious variations of taste, than is exhibited in the responses of our most celebrated professors, on almost every point submitted to their consideration". (225) Young must have considered a challenge to be able to succeed where the most eminent professors, engineers and practitioners have failed. However, his objective was not to exercise a bitter criticism; he saw in the questions many fundamental aspects of bridge design and used them as line of argument to direct the reader to the whole process of bridge design: "It would be useless to dwell on the numerous errors with which many of the answers abound; but the questions will afford us a very convenient clue for directing our attention to such subjects of deliberation as are really likely to occur in a multiplicity of cases; and it will perhaps be possible to find such answers for all of them, as will tend to remove the greater number of the difficulties which have hitherto embarrassed the subject."

In what follows we will examine only those answers directly relevant to arch design and analysis. The complete, numbered, list of questions is given in the Appendix at the end of this paper.

What is structure? Arch or frame behaviour (Question I)

The design presented by Telford is very complex and Question I addresses the first crucial stage in the structural analysis of any building construction: What parts of the work form the structure? In particular, the question makes an explicit division in two ways of structural behaviour: the "arch" (working in compression) and the "frame" (with members either working in compression or in tension).

The answer of Young is extremely lucid. He argues first that the analyst has some freedom in the way to consider the behaviour of the structure, but also that the load tend to follow the paths formed by the more rigid parts of the structure: "there is also a natural principle of adjustment, by which the resistance has a tendency to be thrown where it can best be supported". (225) Then follows a discussion on the functioning of the several arch ribs which can be seen in the design. He concludes that the transmission of the load concentrates in the lower ribs: the upper, flatter, ribs which produce a greater thrust and an slight movement of the buttress will relieve the load from them and transmit it to the lower ribs. It is, then, the lower ribs which transmit the load and it is the lateral thrust produced by them which governs the design, and not the strength of the material which constitutes the arch. Also, the thrust will be less if the load is concentrated in the inferior ribs, and all the circumstances contribute to that "natural adjustment" cited above.

The arch transmits most of the load. The frame may contribute "affording a partial resistance if required . . . . the principal part of the force ought to be concentrated int he lower ribs, not far remote from the intrados". But he remarks

again that the line of thrust, the curve of equilibrium, must not coincide with the intrados (in fact this will produce an overstressing of the arch), nor have to be parallel to it, as it has considered until then.

Finally he relates the nature of the material with the structural type: arches work mainly in compression, and the utility of cast iron lies in his good compressive strength and not in the possibility of connecting different members forming a truss: "the true reason of the utility of cast iron for building bridges, consists not, as has often been supposed, in its capability of being united so as to act like a frame of carpentry, but in the great resistance which it seems to afford to any force tending to crush it".

Curve of equilibrium for dead load (Question III)

Question III is formulated within the frame of the equilibration theory, in which there is a direct relationship between the load and the form of the arch. To discuss the matter in depth, Young says, "would involve the whole theory of bridges" (228) and that he will limit the discussion to the proposed structure, in order to ascertain its strength and, if necessary, to suggest "any alterations . . . compatible with the general outlines of the proposal, to remedy any imperfections which may be discoverable, in the arrangement of the pressure". He is going, then, to make an analysis of the arch ribs, as forming the structure which supports the whole weight of the bridge.

He begins stating that the equilibration theory does not afford a means to analyze the bridge as the distribution of the loads "differ so materially from that which is required for producing an equilibrium in a circular arch of equable curvature" and this has led some experts to consider the whole structure a frame or truss (cf. Fig. 8, above).

Young insists again in what was his main contribution to arch theory, to free the curve of equilibrium from the form of the arch and he states this with utmost clarity: "The truth is, that it is by no means absolutely necessary, nor often perfectly practicable, that the mean curve of equilibrium should agree precisely in its form with the curves limiting the external surfaces of the parts bearing the pressure, especially when they are sufficiently extensive to admit of considerable latitude within the limits of their substance". (229) The arch requires a certain thickness to contain with ease a curve of equilibrium, as its form does not coincide with that of the arch; implicitly Young is here considering a geometrical factor of safety. The problem of the analysis is, then, "to determine the precise situation of the curve of equilibrium in the actual state of the bridge". After this a check should be made relating the safety of the joints "and if this security is not deemed sufficient, the whole arrangement must be altered".

Now Young passes to apply the general Propositions on the equilibrium of arches to analyze Telford's design. He considered all the load concentred in a "typical" plane arch rib of the dimensions stated in Telford's design and supposes that this load has a parabolic form $w = a + bx^2$. From an inspection of the general form of the bridge (and, also probably from the estimations of the weights given by some experts in the Fourth Report though he is not explicit about it) he considers that the load is about three times greater in the abutments as in the crown. Then, for $x = 300$ feet, $w = 3a$ and $90{,}000b = 2a$, so that $b= (1/45{,}000)a$. Substituting this values in equation (8) he obtains the equation of the curve of equilibrium

$$my = \frac{1}{2}ax^2 + \frac{1}{540000}ax^4 \qquad (9)$$

there are two constants m (the horizontal thrust) and a the height of the load at the keystone.

Young considers that the curve of equilibrium should pass through the middle of the keystone and, also, through the middle of the vertical section at the springings. The circular arch of intrados is defined by the span (600 feet) and height (65 feet), and this leads to a radius of 725 feet, with a total angle of aperture of $2 \times 24.45° = 48.9°$. The arch of extrados can be deduced from the drawings in the Fourth Report (Fig. 2, above):taking the middle of the extreme ribs, the thickness at the keystone is 8 feet and at the springings 10. The vertical section at the springings will be a little greater (by a factor (1.08 = 1/cos(22.45°)), but disregarding this, vertical distance between the middle points of both vertical sections will be 64 feet. Of course, Young does not explain all this and only says: "Now the obliquity to the horizon being inconsiderable, this ordinate will not ultimately

Distance x.	Versed sine of the intrados.	Versed sine of the circular arc.	Ordinate y.
50	1.73	1.71	1.34
100	6.94	6.82	5.38
150	15.66	15.43	13.00
200	28.13	27.70	24.50
250	44.42	43.81	41.01
300	65.00	64.00	64.00

Table 1
Ordinates of the intrados, the middle line of the arch and of the curve of equilibrium, calculated by Young for Telford's design for London Bridge (1817).

be much less than the whole height of the arch; and its greatest value may be called 64 feet".

At the springings $x = 300$ and $y = 64$, and substituting in equation (9) we obtain $m/a = 937.5$ feet, which is precisely the radius of curvature of the curve of equilibrium at the crown. (We may compare this value with the radius of the intrados of 725 feet.) Substituting this value again in equation (9) we obtain the expression of the curve of equilibrium:

$$y = \frac{1}{1875} x^2 \left(1 + \frac{1}{270000} x^2\right) \tag{10}$$

Now, Young calculates the ordinates at different points and makes a table to compare the ordinates of the curve of equilibrium with those of the line of intrados and with the middle line (the circle passing through the middle of the keystone and the vertical section at the springings), Table 1.

Young finds a maximum vertical distance of 3.20 feet (the radial distance being nearly 3 feet) between the middle line and the curve of equilibrium at 200 feet from the center, that is, only a little more than one feet apart from the border and this will produce a great compression on this section: ". . . the curve of equilibrium will rise more than 3 feet above its proper place; requiring a great proportion of the pressure to be transferred to the upper ribs, with a considerable loss of strength, for want of a communication approaching more nearly to the direction of the curve". (230) (In fact, if we displace the curve of equilibrium 1.6 feet downwards, this will be the maximum distance from the middle line, which will be almost contained within the middle third of the section, as the vertical thickness will be at this point 8.9 feet. This device is used later in the analysis of Blackfriars bridge.)

Young finds the discordance between the form of the curve of equilibrium and that of the arch excessive and says "it would, however, be much better to have the arch somewhat elliptical in its form, if the load were of necessity such as has been supposed".

Internal forces in the arch, and thrust against the abutments (Question IV)

The question is, again, formulated within the frame of the equilibration theory. If the curve of equilibrium has the form of the intrados then, knowing the load at the keystone the thrust may be calculated directly, but Young remarks that: "It appears from the preceding calculations. that the weight of the 'middle section' alone is not sufficient for determining the pressure in any part of the fabric . . .". (231) But if we know the expression of the curve of equilibrium we may calcu-

late directly its radius of curvature r and the horizontal thrust is (eqn 9) $m = ra$, being a the depth of the load at the crown; "and by combining this thrust with the weight, or with the direction of the curve, the oblique thrust at any part of the arch may be readily found". (231)

Now Young gives a simple procedure to do this. For the case studied (parabolic load), the form of the curve is defined by the form of the load (the relation between a and b) and the points of passage of the curve. The value of a remains undefined. Young, now, set himself to obtain this value. To do this he established the general equilibrium of the half arch: at the springings the thrust must give a vertical component equal to the weight of the half arch, i. e., the tangent of the curve of equilibrium must be equal to W/m, being W the weight of the half arch and m the horizontal thrust. At the abutments $w = a + bx^2 = 3a$, so that $bx^2 = 2a$.

Differentiating the general equation of the curve of equilibrium (eqn. 9, above) we obtain $\dfrac{dy}{dx} = \dfrac{a}{m}x + \dfrac{1}{3}\dfrac{b}{m}x^3$, and at the abutments, $bx^2 = 2a$, $\dfrac{dy}{dx} = \dfrac{5}{3}\dfrac{a}{m}x = \dfrac{5}{3}\dfrac{x}{r}$. For $x = 300$ feet and $r = 937.5$ feet, $dy/dx = 0.5333 = 8/15$, nearly. Therefore, the horizontal thrust at the abutments will be 15/16 of the total weight of the bridge. Now, this weight was estimated by Robison in 10,000 tons (6,500 tons of cast iron, plus the weight of the road), and the horizontal thrust will be $m = 9470$ tons. The load at the keystone will be, then, $a = m/r = 9470/937.5$ or nearly 10 tons. (The surface of the road over the bridge being nearly 18,000 square feet and the total weight of the road 3,500 tons, the superficial load will be 0.20 tons/feet^z, which at the keystone, will lead to a total load $0.20 \times 45 = 9$ tons, the difference being the weight of the ironwork on this place, which is plausible.)

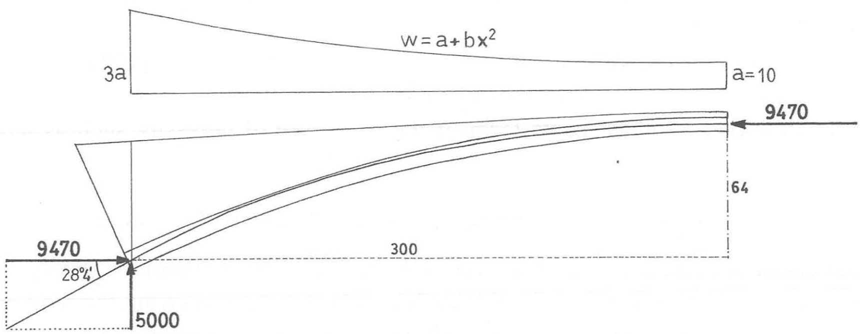


Figure 9
Equilibrium of the arch ring of Telford's design for a dead load of parabolic form. The curve of equilibrium passes through the middle of the joints at the crown and abutments.

Young notices that, though the thrust is greater than the calculated by the equilibration theory applied at the crown (the radius of curvature of the curve of equilibrium is greater than that of the intrados), it is less than will be expected from the inclination of the intrados at the springings, 24°27′ in comparison with the inclination of the curve of equilibrium, atn(8/15) or 28°4′. Another proof of the inconsistency of the old theory.

BLACKFRIARS BRIDGE: Now Young, comes back again to the matter of calculating the curve of equilibrium and takes as an example Blackfriars Bridge, which was considered then one of the best examples of stone bridge. The curve of intrados has three centres (Fig. 10) and the radius of curvature of the central part (4/5 of the span) is 56 feet. Young considers that the continuation of this arch will give very nearly the distribution of the load (the shaded curved triangle ABC in figure 10 is the difference).

Then we are in the case of a load determined by a horizontal extrados and a circular intrados (eqn. 8, above). Now Young determines that the curve should pass through the middle of the keystone, 3 feet above the intrados, and the middle of the vertical section at the springings, which he estimates in 12 feet. If the height of the arch is 40 feet, the height of the curve of equilibrium will be 40 + 3 – 12 = 31 feet. The total thickness at the crown is 6.58 feet (6 feet of the keystone plus 0.58 of the road). Therefore the load will be proportional to this quantity and Young takes $a = 6.58$. Substituting in equation (9), we obtain $my = m$ 31 = 13.510, and then $m = 436$ feet, a quantity proportional to the horizontal thrust. The radius of curvature at the keystone is $r = m/a = 66.25$, i.e., as in Telford's design greater than the radius of the intrados.

Now he calculates the ordinates of the curve of equilibrium, for different values of x and also calculates the ordinates of the middle line of the arch, a circular arc of cord 100 feet and height 31 feet (radius of 55.8 feet, almost the same as

Distance x.	Ordinate y.	Middle of the Arch-stones.
10 feet	.76	.90
20	3.12	3.72
25	5.13	6.12
30	7.71	8.75
40	15.81	16.81
50	31.00	31.00

Table 2
Ordinates of the curve of equilibrium and the middle line of the arch in Blackfriars Bridge. (Young 1817)

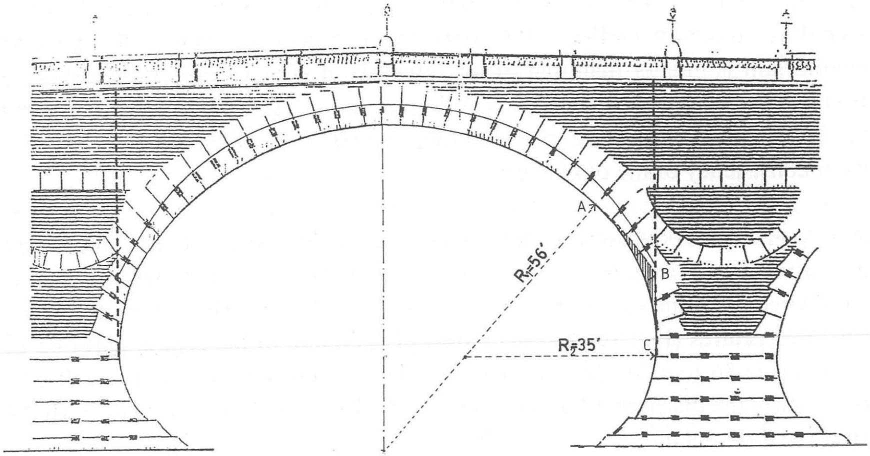


Figure 10
Section of Blackfriars bridge with the curve of equilibrium calculated by Young drawn on it by the author.

that of the intrados). He forms a Table to check the deviation of the curve of equilibrium from the middle line, Table 2: "Hence it appears that the greatest deviation is about 30 feet from the middle, where it amounts to a little more than a foot." (232) At this point, the radial deviation will be nearly $1.04 \times 0.84 = 0.88$, to be compared with a thickness of a little more than 6 feet.

Now Young makes one crucial comment. Until now the curve of equilibrium have had to pass through the middle of the sections at the springings and the keystone. He proposes now to displace downwards the curve of equilibrium half of the vertical distance, so that it will deviate the same quantity at the three critical points: "But if we suppose this deviation divided by a partial displacement of the curve at its extremities . . . it would be only about half as great in all three places; and even this deviation will reduce the strength of the stones to two-thirds, leaving them however still many times stonger than can ever be necessary." Indeed, for a deviation of 0.5 feet and a thickness of 6 feet, the mean stress will be multiplied by a factor $(6 + 6\ (0.5))/6 = 3/2$ (eqn. (2), above), which Young interprets as a reduction of 2/3 of the total strength of the section.

The calculated value of m represents a quantity proportional to the real thrust: ". . . the horizontal thrust is here compressed by $m = 436$, implying the weight of so many square feet of the longitudinal section of the bridge; while, if we determined it from the curvature of the intrados, it would appear to be

only $56a = 368''$. (The calculated thrust being almost 20% greater than the value of the old equilibration theory.) If we call γ the specific weight of the masonry and l the breadth of the bridge, the total thrust will be $436(\gamma l)$.

Young tries in this example to be very minute with every detail and passes to discuss the influence of the direction of the joints and the consideration of the different specific gravities of the materials, but concludes that "so minute a calculation is not necessary in order to show the general distribution of the forces concerned, and the sufficiency of the arrangement for answering all the purposes intended". (233)

Effect of an additional weight placed anywhere over the bridge (Question V)

This is the most difficult question to answer as it implies the analysis of an asymmetrical load. Arch theory has been confined to symmetrical arches and loads until the second half of the XIXth century. This constitutes, again, a challenge to Young as there were no precedents of such an analysis. Young recognizes that a weight placed on the arch will modify the form of the curve of equilibrium: "When a weight is placed on any part of a bridge, the curve of equilibrium must change its situation more or less, according to the magnitude of the weight". Now he affirms, maybe thinking in the analogy with an inverted frame polygon that: "the tangent of its inclination must now be increased by a quantity proportional to the additional pressure to be supported, which, if the weight were placed in the middle of the arch, would always be equal to half of it". To estimate this change of inclination the best way is to find the point where the new curve of equilibrium (dead load plus the additional weight) is horizontal because in this case "the vertical pressure to be supported at each point of the curve must obviously be equal to the weight of the materials interposed between it and this new summit of the curve". (233)

This last observation permits him to locate this point of horizontal thrust. With reference to figure 11, where we have a bridge with a total weigth W which supports an additional load Q located at a distance b from the nearest abutment. Obviously, the vertical reactions will be that shown in the figure and the weight P of the load between the point of horizontal tangent and the keystone is $(b/s)Q$, and therefore:

$$\frac{P}{Q} = \frac{b}{s} \tag{11}$$

Young express this relation as follows: "the distance of the new summit of the curve from the middle must be such, that the weight of materials intercepted be-

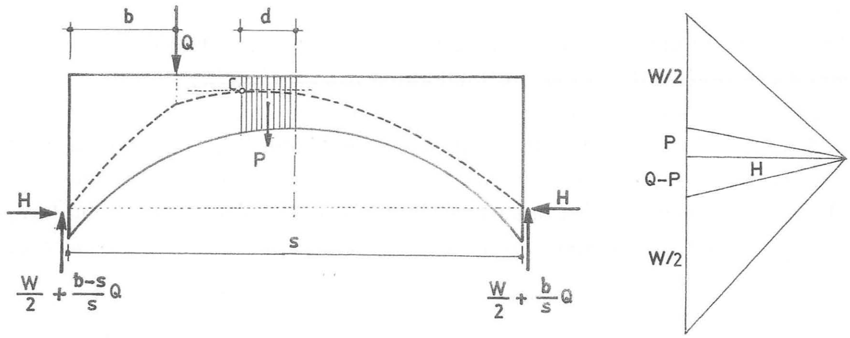


Figure 11
Calculation of the point of horizontal tangent in the curve of equilibrium distorted by the action of an additional weight.

tween it and the middle shall be to the weight as the distance of the weight from the end to the whole span" (234) and gives no demonstration.

Once this point is found "the tangent of the inclination must everywhere be increased or diminished by the tangent of the angle at which the lateral thrust would support the weight of this portion of the materials; except immediately under the weight, where the two portions of the curve will meet in a finite angle, at least if we suppose the weight to be collected in a single point".

Young explains the procedure applying it to Telford's design: "If, for example, a weight of 100 tons, equal to that of about 10 feet of the crown of the arch, be placed half-way between the abutment and the middle; then the vertex of the curve, where the thrust is horizontal, will be removed $2^1/_2$ feet towards the weight." Applying the above formula $(P/100) = (600/150) = 1/4$, then, $P = 100$ tons, which considering the load uniform this distance, which is very nearly true, is equivalent to the weight of 2,5 feet of the load at the crown (which was calculated before as 10 tons), and the horizontal thrust have been calculated as 937.5 feet of the same load. The objective is to deduce the new curve of equilibrium, dead plus additional weight, transforming the curve of equilibrium for the dead load. He explains the procedure in a synthetic way:

> . . . so the tangent of the additional inclination will be 2.5/937.5 = 1/375, and each ordinate of the curve will be increased 1/375 of the absciss, reckoning from the place of the weight to the remoter abutment; but between the weight and the nearest abutment, the additional pressure at each point will be 10 – 2.5 = 7.5 feet, consequently the tangent will be 1/125, and the additions to the ordinates at the abutments will be 450/375

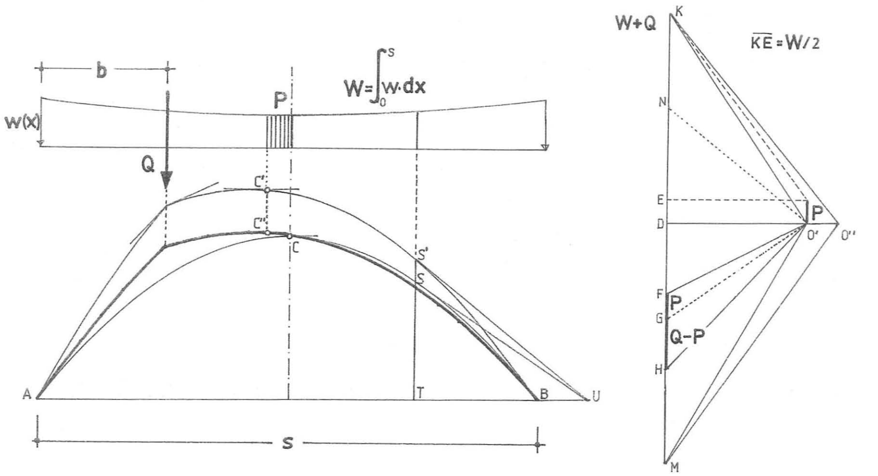


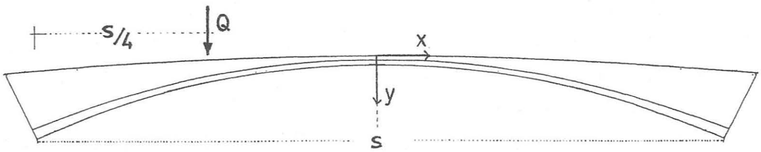
Figure 12
Graphical explanation of Young's method of obtaining the curve of equilibrium for dead plus additional weight, transforming the curve of equilibrium for dead load.

> and 150/125, each equal to 1 (1/5) foot, and at the the summit 150/375 = 2/5, which. being deducted, the true addition to the height of the curve will appear to be 4/5. But the actual height will remain unaltered, since the curve is still supposed to be terminated by the [middle of the] abutments, and to pass through the middle of the key-stone; and we have only to reduce all the ordinates in the proportion of 64.8 to 64.

The procedure is completely correct and shows a enormous ingenuity and is based in the properties of the tangents of the sides of a funicular polygon. figure 12 tries to explain the procedure graphically, with the aid of the force polygon.

Following the method it is very easy to calculate the ordinates of the new curve of equilibrium and to study its deviation from the middle line. In Table 3 we have tabulated the results:

The additional weight is located at $x = -150$ feet from the crown. Young only studies what he considers the critical points: the nearest point of the curve of equilibrium of the dead load, at x = 200 feet, and the point directly under the load. In the first case, " . . . at 200 feet from the summit the ordinate, instead of 24.50 + 200/375 = 25.03, will be 24.72, so that the curve will be brought 2fi inches nearer to the intrados, which, in the proposed fabric, would by no means



x	y_m	y	y_q	$y_m - y$	$y_m - y_q$
–300	64	64	64	0	0
–250	43.82	41.04	40.93	2.77	2.88
–200	27.73	24.49	24.18	3.24	3.54
–150	15.46	13.00	12.43	2.46	3.03
–100	6.83	5.53	5.19	1.30	1.63
–50	1.70	1.34	1.19	0.35	0.50
0	0	0	0	0	0
50	1.70	1.34	1.45	0.35	0.24
100	6.83	5.53	5.72	1.30	1.10
150	15.46	13.00	13.22	2.46	2.24
200	27.73	24.49	24.71	3.24	3.01
250	43.82	41.04	41.19	2.77	2.62
300	64	64	64	0	0

Table 3
Ordinates of different curves and their vertical distances from the middle line of the arch. Origin at the middle point of the keystone: y_m middle line of the arch; y, curve of equilibrium for dead load; y_q, curve of equilibrium dead plus point load.

diminish its strength" (the slight differences in the table are due to the rounding of the calculations). In the second case, the disturbance is greater: " . . . immediately under the weight, the ordinate 13 – 150/375 = 12.6 will be reduced to 12.45, and the curve raised between six and seven inches, which is a change by no means to be neglected in considering the resistances required from each part of the structure". (235) In fact, as may be seen in the Table 3, though the movement of the second point is much greater, the distance of the curve of equilibrium to the limit of the arch is almost the same. The greatest deviation is found 50 feet nearer the abutment ($x = -200$) and is 3.54, equivalent to a radial deviation of 3.7 feet, in a place where the radial thickness is nearly 9 feet.

Now, looking at Table 3 is evident that most of the curve of equilibrium is over the middle line (only some 20 feet apart from the crown on the right side is under it), so, to reduce the stresses Young may have been used the same procedure applied in the analysis of Blackfriars bridge, to displace the line downwards half the greatest distance, i. e. 3.54/2= 1.77. This will be the greatest deviation and the curve of equilibrium will be comfortably within the middle half of the section and only a little outside of the middle-third.

Finally, he stresses that the total thrust increases very little in comparison with the thrust of the dead load only. The problem of the action of an addtional weight is the distortion produced in the curve of equilibrium, not the increase on the thrust.

Best form of the arch and dimensioning of its members (Question VII)

The question refers to the influence of the degree of surbaissement on the thrust and internal stresses and also to the possible advantages of an elliptic profile. Young asnwers to both questions but includes, also, a discussion on the strength of materials and the possible sections of the main ribs.

To discuss the effect of an increase of the height of the arch some assumption as to the variation of the load must be made; Young supposes that the weight remains constant and it is evident that he is thinking in a vertical "stretching" of the original form. In this case the vertical position of the center of gravity does not change and the thrust will diminish in the same ratio as the height grows. A change from 65 to 75 feet height will suppose to pass in the studied curve of equilibrium from 64 to 73 and the thrust will be reduced from 9470 tons to 8300 tons. Being an affine transformation the value of the thrust diminishes, but the relative deviation from the middle line will remain the same: "The additional thrust occasioned by any foreign weight would also be lessened, but not the vertical displacement of the curve derived from its pressure; and since the whole fabric might safely be made somewhat lighter, the lightness would again diminish the strain". (236) This assertion is another proof of the deep grasp of Young on the geometrical properties of the lines of thrust.[5]

Then Young discusses in some detail the problem of the strength of the materials and its role in the design of the main ribs of the arch. He considers that a moderate value of the crushing strength of cast iron is about 50 ton./sq.in. [800 N/mm^2]. The total oblique thrust is 10,730 tons which divided by 50 gives an area of 215 inches and which he multiplies by three to obtain a section of 600 sq.in., which will suppose nearly as many tons of cast iron in the ribs "upon this very low estimate of the strength of cast iron." (237)

In fact, to make cast iron work at one 1/3 of its strength 50/3=16.7 ton/square inc, or 270 N/mm^2, is by no means a "very low estimate". Even for constructive

reasons would have been impossible to divide this area between the thirteen ribs of the design.

Skempton (1980) estimates from the drawings a section of 100 sq. inches per rib, and that will mean 1300 sq. in., and the stress will be less than one half, 10730/1300 = 8.25 tons/sq.in [130 N/mm^2] . However, Skempton notices the reduction of the strength for slender pieces, and comparing with stresses in contemporary buildings and bridges, considers an stress around 4.5 tons/sq.in. as exceptionally high (and that means a coefficient of 1/10 of the crushing strength!). But Young is mainly concerned with supplying the internal forces in the arch. (No doubt, would Telford's design have been accepted, in situ tests of specimens would have been made as was usual with any great iron construction.)

Now Young treats the case of stone bridges and discuss their limit spans: "Calcareous freestone supports about a ton on a square inch [15 N/mm^2], which is equal to the weight of a column not quite 2000 feet [600 m] in height". Young is using here the parameter invented by Perronet and used by Gauthey (Huerta 2004) to measure the crushing strength of a material: the height of a column of uniform section which just collapses at the base $h_l = \sigma_c/\gamma$, where σ_c is the crushing strength and γ the specific weight of the material. (The value of 2000 feet seems very moderate and Rankine (1858) gives this figure for weak sandstone; ordinary sandstone having a double strength and granite five times more, with limit heights of 4000 and 10000 feet, or 1.2 and 3 km respectively.)

Then, he discusses the maximum span which can be attained by stone arches: ". . . consequently an arch of such freestone, of 2000 feet radius, would be crushed by its own weight only, without any further load". In an arch of catenarian form, which supports its own weight the stress at the keystone is $\sigma = r\gamma$, where r is the radius of curvature and the limit radius $r = h_l = \sigma_c/\gamma$. Therefore, ". . . for an arch like that of a bridge, which has other materials to support, 200 feet is the utmost radius that it has been thought prudent to attempt; although a part of the bridge of Neuilly stands, cracked as it is, with a curvature of 250 feet radius; and there is no doubt that a firm structure, well arranged in the beginning, might safely be made much flatter than this, if there were any necessity for it". Young is exhibiting a great confidence in iron and distrust for masonry, an attitude which will grow during the whole XIXth century, but which has no scientific basis.[6]

As for the form of the arch, Young insists in the advantages of the elliptical form, as it adapts itself better to the form of the curve of equilibrium.

Use of scale models (Questions VIII and IX)

It is considered the kind of model to be used in ascertaining the safety of the design and of what size should be built. Young is very clear about the matter: hanging models will permit to check the stability of the arch, but if the model tries to

study the effect of "the cohesion or connection of the parts" the results will be "extremely uncertain".

Young explicits the mode in which the experiment should be made: "the parts corresponding to the blocks of the arch should be formed of their proper thickness and length, and connected with each other and with the abutments by a short joint or hinge in the middle of each, allowing room for a slight degree of angular motion only . . . [and] if the curve underwent no material alteration by the suspension, we should be sure that the calculation was sufficiently correct". If this is not the case, "the arrangement of the materials might be altered". He, then, makes a suggestion to ease the use of the model: " . . . the investigation might be facilitated by allowing the joints or hinges connecting the block to slide a little along their surfaces, within such limits as would be allowable, without too great a reduction of the powers of resistance of the blocks".

There is no drawing, but the text may be interpreted as hanging block model. The size of the blocks calculated in function of their respective weights and the hinges located within the section of the arch and allowing a vertical displacement within it. This interpretation has been represented in figure 13.

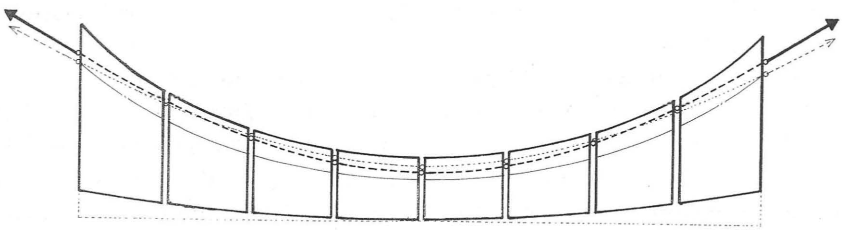


Figure 13
Hypothetical reconstruction of Young's hanging-block model. The hinges can move vertically within the ring of the arch,materializing different curves of equilibrium.

As for the size of the model, he states that it "is of little importance, and it would be unsafe to calculate the strength of the bridge from any general comparison with that of the model".

Design and construction of the abutments (Question XI)

This is a most important question. Flat arches produce a great thrust and, besides, the thrust has a considerable inclination, so that the danger of failure by sliding must be considered. Young, apparently considers the preliminary design of Telford as insufficient and makes a number of suggestions.

Of course, the matter is heavily dependant on the nature of the soil. Young cites the case of St. Saviour's church, built nearby, as a proof of a soil of moderate quality. Besides he considers that, if the foundation rests on piles the sufficient degree of safety may be acquired.

Then proceeds to suggest the general disposition and dimensions of secure abutments in the case of a soft soil, employing piles so that the total weight mobilized to resist the thrust reach 100,000 tons, the main objective being to prevent absolutely a failure by sliding: "When, indeed, the earth is extremely soft, it would be advisable to unite it into one mass for a large extent, perhaps as far as 100 yards in every direction, for such a bridge as that under discussion, by beams radiating from the abutments, resting on short piles, with cross pieces interspersed; since we might combine, in this manner, the effect of a weight of 100,000 tons, which could scarcely ever produce a lateral adhesion of less than 20,000, even if the materials were semifluid" (242)

Then, comments the proper direction of the joints of masonry within the buttress, a matter of enormous importance in the case of surbaissée arches: the masonry should be built with the joints normal to the direction of the line of thrust within the buttress. Finally, he recommends that the piles at the base of the buttress should be driven following the direction of the thrust at the extreme of the curve of equilibrium.

The design of the abutments proposed by Telford has been minutely examined by Skempton (1980). He estimates the weight of the abutments in 63000 tons and calculates that the thrust at the base is well inside the middle third and produces a maximum pressure of 5 tons/sq.ft. [550 kN/m^2]. But, Skempton makes a particular study of the differential settlement at the base of the abutments and gives a table of its evolution. He estimates the final tilt, after the complete consolidation of the soil, in 0.3°, leading to a total spreading of 4 inches. Skempton sees in this a serious inconvenient and cites the case of Staines Bridge "which suffered severe damage and had to be taken down, as the result of a 3 in. movement of one of the abutments. Its span was 181 feet. Once again, then, we find a very uncomfortable feature in the design; especially when it is remembered that the rib stresses would have been exceptionally high even without the yielding of the abutments". (Skempton 1980)

No doubt Skempton calculations of the inclination of the buttresses are correct, but it is difficult to believe that such a tiny movement of the abutments, would have had such an enormous effect as it is supposed to have caused in Staines Bridge. There the displacement was $3/(181 \times 12) = 1/724$ of the span, which looks very moderate; but in London Bridge, it amounts to 1/1800 of the span. The yielding of the buttresses would have produced the typical three-hinge pattern, with a concentration of stress, but it appears that cast iron has sufficient

compressive strength to withstand this effect, in the same way as stone arches have made during centuries or millennia (for the calculation of cast iron arches, see Heyman 1982). Young's modifications would have reduced notably the value calculated by Skempton.

Possible improvements and safety of the design (Questions XIX and XX)

The rest of the questions address mainly practical matters: the construction of the scaffolding, the type of iron to be used, the possibility of casting the members with sufficient precision, the size of the castings, the use of "iron cement", etc. However, questions XIX and XX imply a summary of the main arguments and will be examined.

Young suggest to eliminate the upper flatter ribs and reinforce the lower ribs forming the arch and, also, "made either in the form of blocks or of frames with diagonals" (245) (following presumably the model employed by Telford in his iron bridges after Bonar Bridge). The profile of the ribs should adjust better to the form of the curve of equilibrium.

Then he treats in some detail the problem of decentering, closely related with the apparition of cracks and concentrations of stress: "It would be necessary to wedge the whole structure very firmly together before the removal of the centres", following a method similar as that employed for stone bridges and which is intended "to enable the stones to bear fully on each other, and which has been very properly adopted in the best modern works". (All this precautions, leading to a certain pre-compression of the voussoirs, may lead to a diminution of the descent of the crown. Another traditional device, which was applied to flat arches and vaults, was to built the arch or vault with an initial stilt so that after deformation will take the desired profile.)

As for the feasibility of the design, Young expresses no doubt about it, and, in fact, he has given the theory and practical calculation tools developments to make all the necessary analysis and corrections of the original design. He insists, again, that the main problem is in the design of the abutments: "The only reasonable doubt relates to the abutments; and with the precautions which have been already mentioned in the answer to the 11th question, there would be no insuperable difficulty in making the abutments sufficiently firm."

Analysis of other "modern" bridges: Southwark and Waterloo Bridges

As has been mentioned in the version of the article "Bridge" printed in the *Miscellaneous papers*, tha last section of the original article, with the title "Modern History of Bridges", was completely suppressed. In fact, only the first part of the section is a brief history of the first iron bridges constructed. The second part

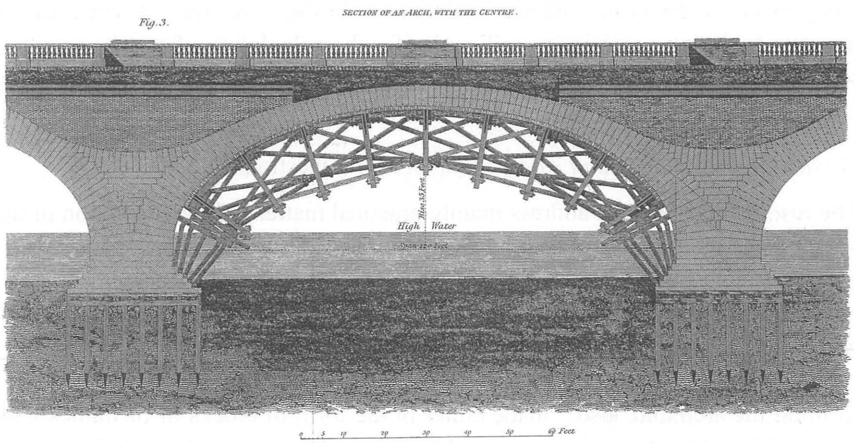


Figure 14
Waterloo Bridge. The curve of equilibrium as been drawn on the right hand half arch. It is the first time that a line of thrust is drawn to explain the stability of a real bridge. (Young 1817)

contains the application of Young's theory of the arch to the analysis of two important bridges: Southwark and Waterloo bridges. This suppression is a grave affair as some parts of the article are difficult, if not impossible, to understand without looking at the calculations made above the cited bridges. Also, the section confirms the main objective of Young: to provide a theory of arches directly applicable to the analysis and design of actual bridges. There is no space here to discuss the ingenuity with which he applied his own theory to the analysis of real bridges. But maybe a good manner to finish this paper is with, perhaps, the first drawing of a line of thrust within an arch bridge, in figure 14.

Conclusions

1. Thomas Young has a deep understanding of the concept of "line of thrust", which he called curve of equilibrium. He formulated and employed this concept nearly twenty years before other authors.
2. He was the first to free this curve from the "straitjacket" of the intrados, the curve depending on the load distribution and expressing the infinite possible equilibrium situations in which an arch may transmit its loads.
3. For arches of stone or cast iron, materials with good compressive strength but low tensile strength the curve of equilibrium must lie within the sub-

stance of the arch, with some geometrical safety, i.e. the curve should not approach too much to the borders.

4. Young obtained the general mathematical expression of the curve of equilibrium for different types of loads (defined by the curves of intrados and extrados), with a view to its application in Bridge analysis. His use of a simple parabolic load is remarkable for its simplicity and applicablity in most cases.
5. For the first time he considered the influence on the stability of the arch of a point load placed anywhere on the extrados. He devised a completely original method of obtaining the corresponding curve of equilibrium transforming that of the dead load.
6. He had clear ideas of the transformation of curves of equilibrium (i.e. the affinity between them) and knew, for example, that the thrust, for a certain load distribution, is inversely proportional to the height of the curve, that the relative vertical distances remain constant, etc.
7. Young's theory of the arch, based on a correct application of the approach of the equilibrium with due respect to the material properties, was well ahead of his contemporaries. The analysis of unsymmetrical arches were made only in the second half of the XIXth century. It apparently had no influence in later writers.
8. Young's analysis of Telford's design is completely correct, combining statements of equilibrium (curves of equilibrium for the given loads) with statements about the material (cast iron must work in compression; therefore the curve of equilibrium must lie within the arch). The notion of a geometrical factor of safety is implicit in many of his statements.
9. Young's approach is within the frame of modern limit analysis and of the Fundamental Safe Theorem. The main corollary of this Theorem is the "approach of equilibrium" and this is precisely the procedure of Young, which he followed with deep understanding.
10. Telford's design, with some modifications, would have been a completely safe structure. Would it have been built, it will be today a symbol of London, in the same way as the Eiffel tower is a symbol of Paris. It is to regret that ignorance, fear and parsimony stopped Telford's grand design.

Appendix

Questions respecting the construction of a cast iron Bridge, of a single arch, 600 feet in the span, and 65 feet rise.

I. What parts of the bridge should be considered as wedges, which act on each other by gravity and pressure, and what parts as weight, acting by gravity only, similar to

the walls and other loading, usually erected upon the arches of stone bridges? Or does the whole act as one frame of iron, which can only be destroyed by crushing its parts?

II. Whether the strength of the arch is affected. and in what manner, by the proposed increase of its width towards the two extremities or abutments, when considered vertically and horizontally? And if so, what form should the bridge gradually acquire?

III. In what proportion should the weigth be distributed from the centre to the abutments, to make the arch uniformly strong?

IV. What pressure will each part of the bridge receive, supposing it divided into any given number of equal sections, the weight of the middle section being given? And on what parts, and with what force, will the whole act upon the abutments?

V. What additional weight will the bridge sustain, and what will be the effect of a given weight placed upon any of the before-mentioned sections?

VI. Supposing the bridge executed in the best manner, What horizontal force will it require, when applied to any particular part, to overturn it, or press it out of the vertical plane?

VII. Supposing the span of the arch to remain the same, and to spring ten feet lower, What additional strength would it give the bridge? Or, making the strength the same, What saving may be made of the materials? Or, if, instead of a circular arch, as in the plates and drawings, the bridge should be made in the form of an elliptical arch, What would be the difference in effect, as to strength, duration, convenience, and expenses?

VIII. Is it necessary or advisable to have a model made of the proposed bridge, or any part of it, in cast iron? If so, what are the objects to which the experiments should be directed; to the equilibration only, or to the cohesion of the several parts, or to both united, as they will occur in the intended bridge?

IX. Of what size ought the model to be made, and what relative proportions will experiments, made on the model, bear to the bridge when executed?

X. By what means may ships be best directed in the middle stream, or prevented from driving to the side, and striking the arch; and what would be the consequence of such a stroke?

XI. The weight and lateral pressure of the bridge being given, can abutments be made in the proposed situation for London Bridge, to resist that pressure?

XII. The weight and lateral pressure of the bridge being given, can a centre or scaffolding be erected over the river sufficient to carry the arch without obstructing the vessels which at present navigate that part?

XIII. Whether would it be most advisable to make the bridge of cast and wrought iron combined, or of cast iron only? And if of the latter, Whether of the hard white metal, or of the soft grey metal, or of gun metal?

XIV. Of what dimensions ought the several members of the iron work work to be, to give the bridge sufficient strength?

XV. Can frames of cast iron be made sufficiently correct to compose an arch of the form and dimensions shown in the drawings, so as to take an egual bearing as one frame, the several parts being connected by diagonal braces, and joined by an iron cement, or other substance?

XVI. Instead of casting the ribs in frames of considerable length and breadth, would it be more advisable to cast each member of the ribs in separate pieces of considerable lengths, connecting them together by diagonal braces, both horizontally and vertically?

XVII. Can an iron cement be made, which shall become hard and durable, or can liquid iron be poured into the joints?

XVIII. Would lead be better to use in the whole or any part of the joints?

XIX. Can any improvements be made in the plan, so as to render it more substantial and durable, and less expensive; And if so, what are these improvements?

XX. Upon considering the whole circumstances of the case, agreable to the Resolutions of the Committee, as stated at the conclusion of their Third Report, is it your opinion that an arch of 600 feet in the span, as expressed in the drawings produced by Messrs. Telford and Douglas, or the same plan, with any improvement you may be so good as to point out, is practicable and advisable, and capable of being made a durable edifice?

XXI. Does the estimate, communicated herewith, according to your judgement, greatly exceed or fall short of the probable expense of executing the plan proposed: specifying the general grounds of your opinion?

Acknowledgements

This article is a revised and expanded version of the draft of the Keynote Lecture of the same title delivered at the *4th International Conference on Arch Bridges* (Barcelona, November 17th–19th, 2004), by invitation of Prof. Pere Roca. The death of my mother impeded me to complete the draft in time to be included in the Proceedings. This paper is dedicated to her memory.

Notes

1. Though the project was signed by Telford and Douglass the credit for the design should be given to Telford. Ruddock (1979, 158) affirms that ". . . all the records convey the impression that he himself made the designs and estimates for London Bridge and that he conducted most of the subsequent investigations and negotiations". For Skempton (1980, 67) ". . . Douglass, though a clever and ambitious engineer. . ., had no experience and probably little knowledge of bridges".
2. The list, as it appears in the Fourth Report, is: Dr. Nevil Maskelyne (the Astronomer Royal), Rev. A. Robertson (Savilian Professor of Geometry, Oxford), Playfair (Professor of Mathematics, Edinburgh), John Robison (Professor of Natural Philosophy, Edinburgh), Dr. Milner, Dr. Charles Hutton (Royal Military Academy, Woolwich), Mr.

Atwood, Colonel Twiss (Woolwich), Mr. William Jessop, Mr. J. Rennie, Mr. James Watt, Mr. John Southern, Mr. William Reynolds, Mr. John Wilkinson, Mr. Charles Bage, General Bentham (Inspector General of the Naval Works of the Admiralty), and Mr T. Wilson.

3. Young considered the article "Bridge" one of his major contributions to the Encyclopaedia Britannica. This is already evident in his correspondence with the editor Napier (Wood and Oldham 1954, 259). But in the list of the 62 articles written for the Encyclopaedia which he included in the catalogue of works of his own autobiography, only three appear in capital letters, Bridge, Egypt and Tides (Hilts 1978, 259), as a sign of their importance.
4. The exposition is very similar to the most important paper of Moseley (1838) on the subject. Moseley formulated all his theory of lines of thrusts (*lines of resistances*) as if there was no precedent. He should have been aware of Young's work.
5. The application of affine transformations to the study of the equilibrium of arches is a powerful tool, which was exploited extensively by Rankine (1858). For a historical study of this approach see Huerta (2004, 407).
6. At the beginning of the XXth century several masonry bridges of more than 300 feet were built; the greatest, in unreinforced concrete, at Cruseilles, 1928, with 140 m or 450 feet. (Huerta 2004, 407).

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Grace and law: The spatial framework from Föppl to Mengeringhausen

Karl-Eugen Kurrer

"Architecture is frozen music". This phrase coined by the philosopher Arthur Schopenhauer (1788–1860) is often quoted in the relevant literature. If it also applies to structural engineering objects, the design work of structural engineers can be regarded as structural composition. This article elaborates this proposition and illustrates it using the example of the historic development of spatial frameworks.

The five platonic bodies

Plato (427–348 BC) divided human cognition into sensibility, intellect and reason. However, sensations, perceptions and notions can never reach beyond the remit of subjective thinking and can never create knowledge. According to Plato, external objects can only be reached through cognition-obscuring sensibility. The light of the mathematical law does not shine in external objects, but in ourselves —after all, it is already contained in the immortal and often risen soul that has seen all things on Earth and in Hades. According to Plato "searching and learning . . . invariably involves recollection" (Seidel 1980, 220).

Mathematical intellect deals with numbers and numerical proportion, as expressed in geometry, for example —leading to true knowledge. Even the computer-drawn rectangular triangles (figure 1 a), from which (in *Timaios*) Plato assembled the regular polyhedrons named after him (Plato 1994, 55–57), would only be imperfect images of an eternal idea that is quiescent in itself, and in which only the geometric triangle theorems could claim to be real. Of particular

interest is Plato's implicit mathematical structural law. Since each triangle consists of rectangular triangles, he selects two triangles from each set: "of the two triangles the equal-sided only comes in one shape (figure 1 a/right), while the unequal triangle can have countless shapes. From these countless shapes we have to select the most beautiful one . . . From . . . the many triangles we regard that as the most beautiful from which the equilateral triangle was formed . . ." (figure 1 a/left) (Plato 1994, 55–56). From the "most beautiful", i. e. the unequal rectangular triangle (figure 1a/left) Plato constructed

- the tetrahedron, consisting of 24 basic triangles, defining the element fire;
- the octahedron, consisting of 48 basic triangles, defining the element air;
- and the icosahedron, consisting of 120 basic triangles, defining the element water.

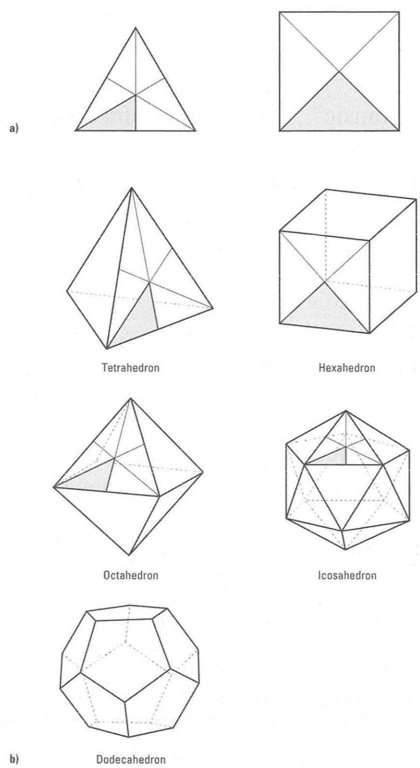


Figure 1
Plato's implicit mathematical construction law; a) Base triangles, b) the five Platonic bodies (Falter 199, 49)

The hexahedron, representing the element earth, is formed from 24 equal-sided rectangular triangles (figure 1 a/right). This harmony of the elements is somewhat disrupted by the dodecahedron (figure 1 b), which Plato was unable to reduce back to basic triangles: "so God used it for the world as a whole" (Plato 1994, 57).

In Plato's explanation of the transition between the four elements the meaningfulness of his polyhedron idea already begins to fade and soon becomes lost in the darkness of his description of nature in *Timaios*. Only his polyhedron idea prevailed, as indicated by Johannes Kepler's (1571–1630) representation of the world order in his *Mysterium Cosmographicum*, published in 1596. Kepler interpreted the proportions of the distances of the planets from the sun through nested Platonic bodies. In 1758, Leonhard Euler's (1707–1783) polyhedron theorem,

$$e + f = k + 2 \tag{1}$$

with e = number of corners, f = number of surfaces and k = number of edges, tied Plato's non-conforming dodecahedron down from the heavenly spheres to the Procrustean bed of mathematical law. While desperately trying to find proof of the existence of God, the religious Euler thus expelled God from the paradise of the Platonic bodies.

Grace and Law

In his work *Politeia*, Plato placed art after science and measured it based on the beauty as the good aspect benefiting the state, and not on the skilfulness of the artists or the imitation of the existing. In contrast, his great disciple Aristotle (384–322 BC), who later taught Alexander the Great, took up the cudgels for art. After all, *techne* is the quintessence of all human skills for achievement: through work, craftsmanship and skilfulness. Aristotle thus moved action in technology, art and science into the centre of philosophising for the first time. Since the drawing is the language of the engineer, Plato's tetrahedron is used to illustrate important terms. Figure 2 (a) uses a tetrahedron to illustrate the interaction of the four realms of nature, technology, art and science:

- technology and art are linked through composition;
- technology and science are linked through modelling;
- art imitates nature;
- science comprehends art and
- recognises nature.

Nature, science and art form the basic triangle, the basis of modern technology.

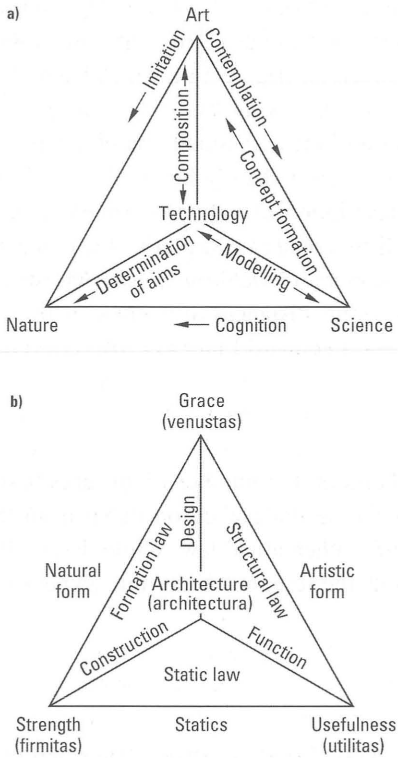


Figure 2
a) The tetrahedron of the four empires and b) the tetrahedron of architecture

There is an objective link between static law, formation law and structural law, figure 2 (b):

- Static law is enshrined by the points of architecture, usefulness and strength, and by the edges of function, statics and construction.
- Formation law is enshrined by the points of architecture, strength and grace, and by the edges of construction, natural form and design.
- Structural law is enshrined by the points of architecture, grace and usefulness, and by the edges of design, artistic form and function.

Strength, statics, usefulness, artistic form, grace and natural form make up the basic triangle, the basis of architecture.

Static law interacts with formation law via construction. Function links static law with structural law. The latter is linked with formation law via design. In spatial frameworks, static law, formation law and structural law also form a higher unit in the shape of the composition law for spatial frameworks, as recognised and implemented in practice by Max Mengeringhausen (1903–1988).

Structural law

Plato had already introduced an implicit mathematical structural law by assembling the tetrahedron, hexahedron, octahedron and icosahedron from two basic triangles (Fig. 1). In 1940 Mengeringhausen formulated eight structural laws for spatial frameworks, based on the above-mentioned polyhedrons (Fig. 18). From a methodical point of view, his first structural law is reminiscent of Plato: *"Load-bearing spatial structures* (spatial frameworks) are ideally composed from equilateral and (or) equal-sided rectangular *triangles* such that *regular multiples* (polyhedrons) in the form of tetrahedrons, cubes, octahedrons, truncated octahedrons, truncated cubes or shapes derived from them are created" (Mengeringhausen 1983, 114).

Figure 3a illustrates Mengeringhausen's first structural law. The basic tetrahedron with an edge length of 1/m (Fig. 3 a/left) can be stacked to form a spatial framework such that a space is created that is completely filled and bounded by tetrahedrons (Fig. 3 a/right). Normalised to 1, mathematically this space is characterised by four families of coordinate planes that intersect in straight lines and points in such a way that each point is defined by four coordinate numbers, the sum of which is always 1. A spatial framework is formed if the straight lines are materialised to members and the points to joints. In 1971 such a spatial framework formed the basis of the Siemens pavilion at the German industry exhibition in São Paulo, Brazil (Fig. 4), although the framework formed by Mengeringhausen's first structural law (Fig. 3 a/right) was not filled in, resulting in three two-layer tetrahedron frameworks.

Static law

In its most basic form, the static law for spatial frameworks manifests itself as a free equilibrium system in a closed tetrahedron framework that is subject to tensile forces F in the four symmetry axes and "responds" with six internal tensile forces S in the members (Fig. 3 b).

Formation law

In composite spatial frameworks such as that at São Paulo (Fig. 4), such an equilibrium system is unlikely to form in practice. The simplest formation law for spatial frameworks illustrates this (Fig. 3 c). It states that, based on the stably

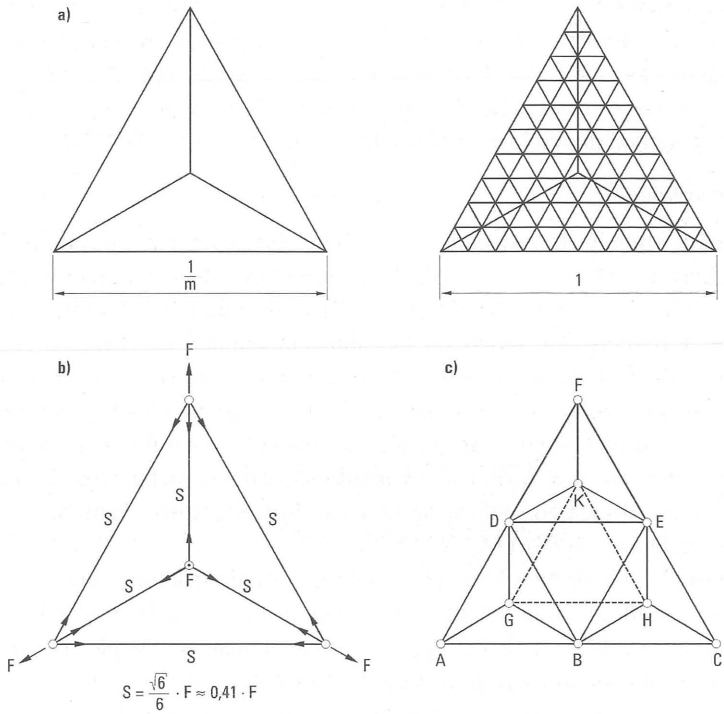


Figure 3
Elements of the law of composition, illustrated using the example of the tetrahedron framework; a) Mengeringhausen's first structural law, b) static law of equilibrium, and c) first formation law for spatial frameworks

supported ball joints of basic triangle ABD formed by three members joint G can be connected stably, as long as G does not lie in a plane formed by basic triangle ABD. The same procedure can be applied to basic triangles BEC and DEF, resulting in three independent stable tetrahedron frameworks with vertices G, H and K. If additional connections are introduced between high points G, H and K, an octahedron filling the space between the three tetrahedrons is created. The whole spatial framework thus created from three tetrahedrons is statically indeterminate to the third degree and can only be calculated by solving the elasticity equations. This play with the formation law can be continued until a spatial framework as shown in figure 4 has emerged. Ultimately there will be n redundant members, requiring n elasticity equations with n unknown member forces to

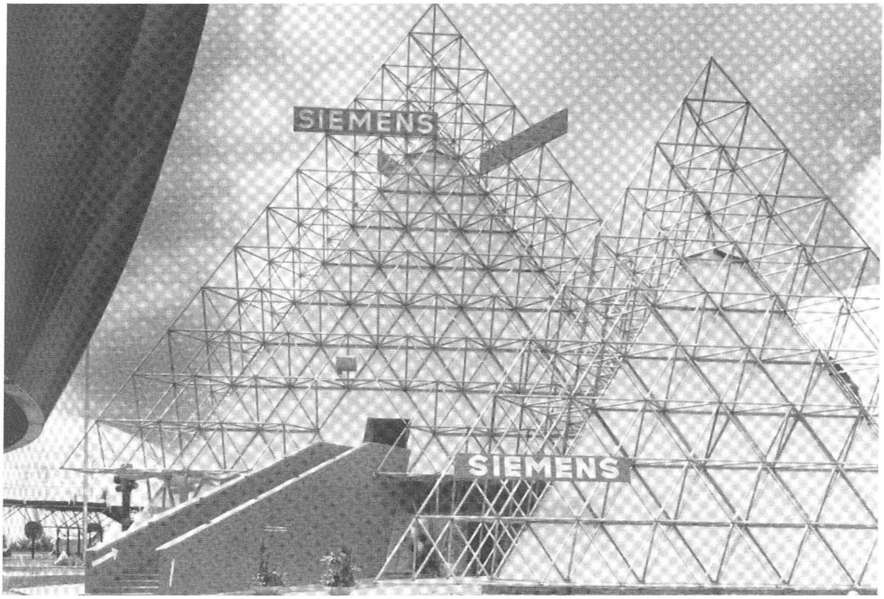


Figure 4
Siemens exhibition pavilion in the shape of a tetrahedron at the German Industry Exhibition at São Paulo in 1971 (Mengeringhausen 1975, 37)

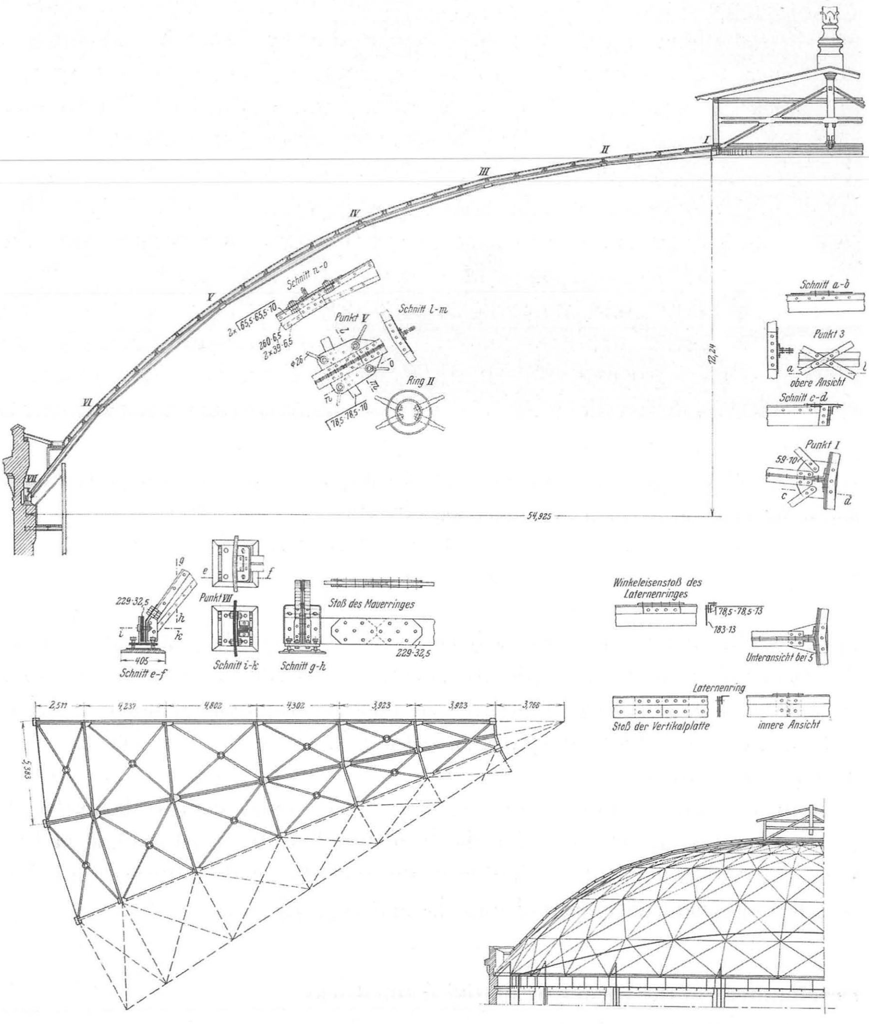
be solved. Since n will be hopelessly large, calculation of the n statically indeterminate member forces using the manual means of classic structural theory is impossible in practice. The reason for the nonconcurrent development of theory and realisation of spatial frameworks that are statically undetermined to a high degree is thus obvious.

In the construction of spatial frameworks grace and law express themselves in the composition law, consisting of the formation law, the structural law, and the static law. But how has the cognition of the static law and the formation law of spatial frameworks developed historically and logically?

Development of the theory of spatial frameworks

Until the last decade of the 19th century, the development of trussed framework theory, which started in 1850, was limited to planar systems. During this decade, classic structural theory also took on its final form as a consistent theory of statically determinate and indeterminate planar elastic frames (Kurrer 2002, 230–257).

Spatial structural systems for factory buildings, railway stations or bridges had an orthogonal structure, so that reduction to planar systems was usually sufficient. In addition, since the beginning of the 19th century spatial engineering thinking had been based on orthogonal projections in the shape of technical



Gasbehälter der Städtischen Gasanstalt in der Fichtestraße in Berlin (erbaut 1875).
Kuppeldurchmesser 54,9 m. Eisengewicht 68 t = 28,7 kg/m².

Figure 5
Schwedler dome above the gasometer of the municipal gas works at Fichtestraße, Berlin-Kreuzberg, constructed in 1875 (Hertwig 1930, plate 7)

drawings, a technique that dominated descriptive geometry. Early spatial structural analysis approaches such as the forward-looking two-volume *Lehrbuch der Statik* (statics textbook) (Möbius 1837) by August Ferdinand Möbius (1790–1868) or Otto Mohr's (1835–1918) article on the composition of spatial forces (Mohr 1876) attracted little interest from structural engineers.

Even for domed structures spatial load transfer was initially not considered, and the radially arranged trusses were analysed using planar statics. One such dome, i. e. that covering the gasometer at Imperial-Continental-Gas-Association in Berlin (Hellweg no. 8) collapsed during assembly in 1860. The engineer responsible for the project, Johann Wilhelm Schwedler (1823–1894), improved the design for the dome structure that was rebuilt one year later, although he still used the conventional technique. In 1863 he designed another dome structure for the same client, covering the gasometer at Holzmarktstraße 28, Berlin, and became the first engineer to make the transition to a dome with a spatial function, which became known as the *Schwedler dome* in the technical literature. Three years he described five further Schwedler domes in *Zeitschrift für Bauwesen*, providing not only the theory behind them, but also a simplified structural calculation technique (Schwedler 1866): Schwedler developed the membrane theory for axisymmetric shells under load and calculated the membrane forces acting on the meridians and parallels. Figure 5 shows the roof of the municipal gasworks at Fichtestraße, Berlin-Kreuzberg, constructed in 1875 in the shape of a Schwedler dome. The iron structure with a dome diameter of 54.9 m and a rise of 12.2 m still exists today. The spatial system is statically indeterminate to a high degree and cannot be calculated in practice using classic member statics. Schwedler therefore had to "blur" his dome to form a two-dimensionally curved elastic continuum and then discretisise it —a technique that August Föppl (1854–1924) methodised to cover other shell-type trussed frameworks (Föppl 1892) and that only became obsolete at the end of the 1960s when computers started to be used for analysing spatial frameworks. Until the third decade of the last century, spatial framework theory developed in three directions:

- Static-structural development of spatial frameworks that could be calculated in practice using classic structural theory. A prominent example is the statically determinate dome over the plenary chamber of the Reichstag building in Berlin (Fig. 6), invented and calculated in 1889 by Hermann Zimmermann (1845–1935), i. e. the *Zimmermann dome*.
- Development of the theory of spatial frameworks and design of shell-type spatial frameworks by August Föppl, following Schwedler's dome system of 1892.

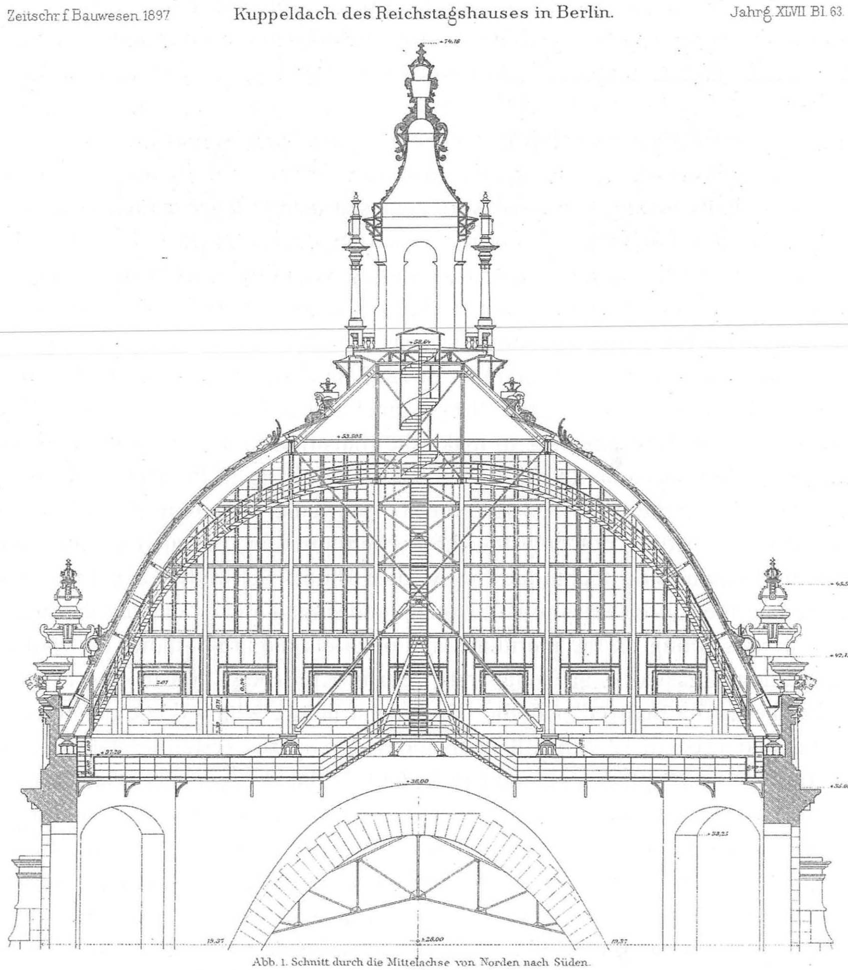
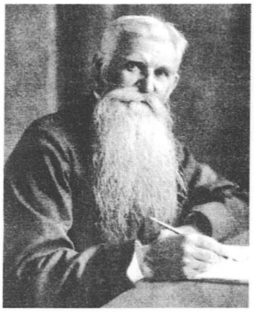


Figure 6
Reichstag dome constructed in 1890 according to Zimmermann's spatial framework concept (Lodemann 1897, atlas, sheet 63)

- Integration of spatial framework theory into classic structural theory by Heinrich Müller-Breslau (1851–1925) and others.

Figure 7 shows portraits of the personalities who made significant contributions to the development of spatial framework theory.



Hermann Zimmermann
(1845–1935)

August Föppl
(1854–1924)

Heinrich Müller-Breslau
(1851–1925)

Otto Mohr
(1835–1918)

Lebrecht Henneberg
(1850–1933)

Wilhelm Schlink
(1875–1968)

Figure 7
The creators of the theory of spatial frameworks

The "Reichstagskuppel" (Dome of the Reichstag building)

On Wednesday, 17 February 1993, an article on the competition for the conversion of the Reichstag building into the German Bundestag appeared in the Feuilleton section of one of Berlin's daily newspapers, *Der Tagesspiegel*. The results of the competition were announced two days later by the then President of the Bundestag, Rita Süssmuth. The article was entitled "Wallots Kuppel kommt nicht wieder" (No comeback for Wallot's dome (Schulz 1993, 19). The author of this paper replied with a reader's letter, stating: "Wallot's dome cannot come back, since it was never built. If the correspondent was referring to the Reichstag dome, he should have written: ‚No comeback for Zimmermann's dome' . . . Who knows the creator of the Reichstag dome? What was Zimmermann's contribution to engineering?" (Kurrer 1993, II).

The author went on to explain the structural wit of Zimmermann's spatial framework invention, which in 1889/90 helped architect Paul Wallot out of a predicament and was probably the reason for Zimmermann's promotion one and a half years later to become the top construction officer in the Prussian ministry for public works. The structural wit, i. e. the formation law and the static law of the Zimmermann dome, comprises the following aspects (Fig. 8):

- The vertical forces A_I to A_{IV} and B_I to B_{IV} are transferred via the four supports in the corners of the drum masonry with rectangular plan and external dimensions of 38.74 × 34.73 m, figure 8 (a).
- The lateral forces are controlled via a sophisticated support system in such a way that the drum masonry can develop its structural plate effect, i.e. in the direction of the wall only shear forces T are present: Zimmermann's spatial framework thus resolved the issue of orthogonal horizontal thrust, which master builders had dreaded for more than 2,000 years.
- The spatial framework has four members in its upper ring, twelve in the lower, twelve in the wall members, eight vertical and four horizontal support reactions acting parallel to the drum masonry, i. e. a total of 40 unknown force parameters, figure 8 (b). On the other hand, for each the 12 joints three equilibrium conditions, i. e. a total of 36 equilibrium conditions, can be formulated. Through special arrangement of the four supports in a and b four further equilibrium conditions can be created, so that the 40 unknown member forces of the spatial framework can be determined solely based on the 40 equilibrium conditions, i. e. the Zimmermann dome is statically determinate.

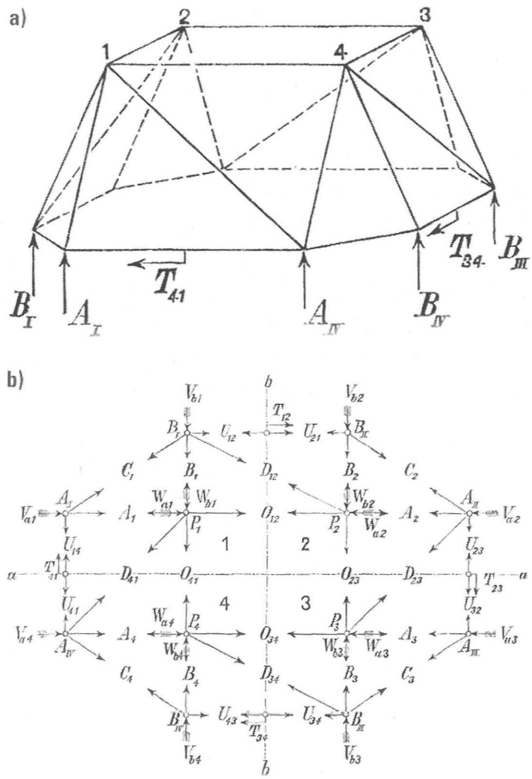


Figure 8
a) Isometric view of the structural system of the Zimmermann dome (Zimmermann 1901, 6).
b) Top view of the support and member forces (Zimmermann 1901, 9)

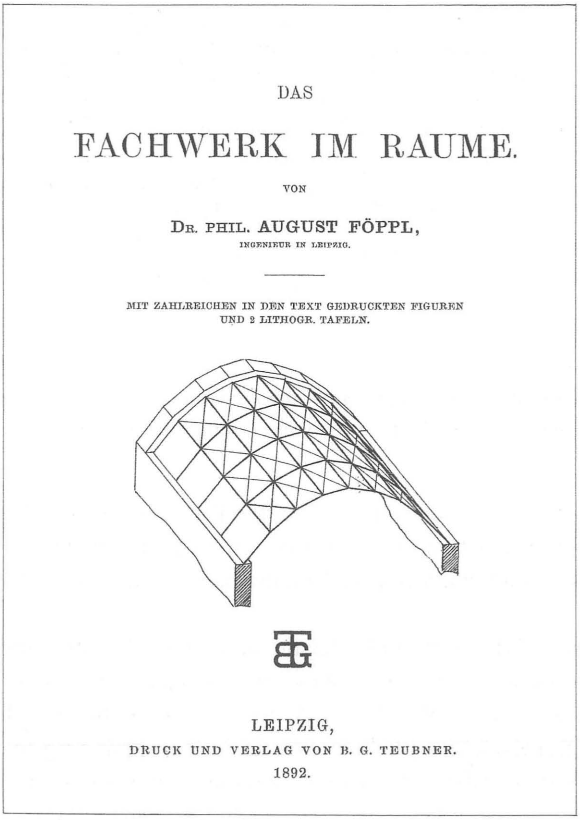
Just like his predecessor Schwedler in the Prussian ministry for public works, Zimmermann still mastered the construction language of the mechanical engineer and the structural engineer: the Zimmermann dome is an ingenious structural machine.

Zimmermann didn't reveal his elegant structural calculation until 11 years after the construction of his dome above the plenary chamber of the Reichstag building (Zimmermann 1901, 1–46). At the same time, he extended his concept from spatial frameworks with four corners in the upper ring to spatial frameworks with any number of corners (Zimmermann 1901, 46–67) and also described derived spatial framework types (Zimmermann 1901, 67–93). Through structuring of the elimination process for the 40 equilibrium equations he succeeded in demonstrating the relationship with the topology of this class of

spatial frameworks with rare clarity: Static law and formation law merge on a mathematically clear level by means of concrete spatial structural forms. Zimmermann thus conveys "aesthetic eigenvalue" to structural calculations.

Foundation of the theory of spatial frameworks by August Föppl

August Föppl's book *Das Fachwerk im Raume* (spatial frameworks) (Fig. 9), published in 1892, was the fruit of his synthetic research of the structural law, the static law and the formation law of spatial frameworks in the shape of the composition law for spatial frameworks, which he had started in 1880 (Föppl 1880). Almost 50 years later Mengeringhausen used this work, which remained unsurpassed until the 1960s, as the basis for his work on spatial frameworks. The title page of Föppl's book showed a trussed shell curved in one direction (Fig. 9),



DAS

FACHWERK IM RAUME.

VON

DR. PHIL. AUGUST FÖPPL,

INGENIEUR IN LEIPZIG.

MIT ZAHLREICHEN IN DEN TEXT GEDRUCKTEN FIGUREN UND 2 LITHOGR. TAFELN.

LEIPZIG,

DRUCK UND VERLAG VON B. G. TEUBNER.

1892.

Figure 9
Title page of the first monograph about spatial frameworks (Föppl 1892)

which he called *"Flechtwerk"*. In 1890 the imperial patent office in Berlin denied it any invention value, claiming that common gasometer guide frames also represent trussed shells (Föppl 1892, 56).

In 1883 Föppl suggested a new domed roof system (i. e. the network dome), the calculation method for which he published on 5 May 1888 in *Schweizerische Bauzeitung* (Föppl 1888). After five-year break he had thus resumed his research on the theory of spatial frameworks. Föppl was prompted by Hacker's recently published work (Hacker 1888), in which he consistently assembled the Schwedler dome from basic triangles and determined the member forces without using membrane theory (even for asymmetric loads) solely with the aid of the joint equilibrium conditions of the trussed framework. On 28 March 1891 Föppl reported on the first application of the network dome for the roof of the central market hall in Leipzig (Fig. 10). This network dome with a scalene pentagon plan and a span of approx. 20 m is 6.80 m high and transfers the loads via five 4.40 m high framework plates into five fixed bearing hinges. The structural system con-

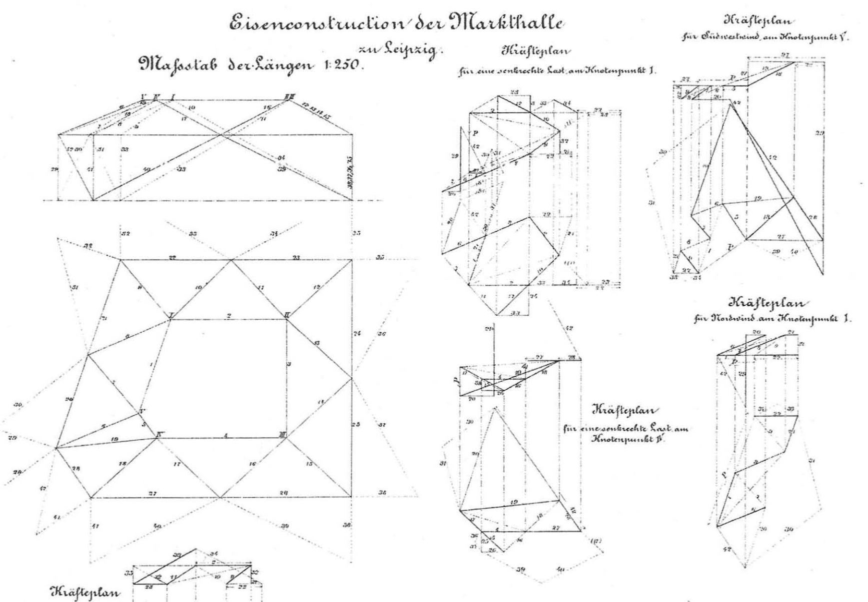


Figure 10
Föppl's graphostatic examination of the network dome of the central market hall at Leipzig (Föppl 1891b, plate)

sists of 42 members with 42 unknown member forces, 14 free joints, each with three equilibrium conditions, i. e. a total of $14 \times 3 = 42$ joint equilibrium conditions. The Leipzig network dome is thus statically determinate, so that the 42 unknown member forces can be determined graphostatically through force diagrams.

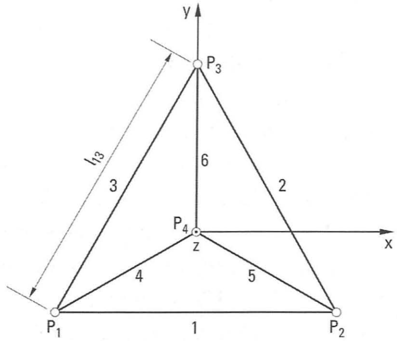
On 14 June 1891 the collapse of the trussed railway bridge over the river Birs at Mönchenstein (today: Münchenstein) near Basel shocked the world. 73 people died and more than 100 others were injured. During the second half of 1891 and 1892, *Schweizerische Bauzeitung* published articles on this bridge disaster on an almost weekly basis. Föppl was the first specialist from abroad to join the discussion on the causes of the collapse on 18 July 1891: "The bridge collapsed, because —as a spatial trussed framework— it was unstable". He concluded that is was unjustified to limit framework theory to planar trussed frameworks. According to Föppl, textbooks only mention spatial trussed frameworks in passing, if at all, and in the past spatial framework theory had largely been ignored. This neglect led to the Mönchenstein disaster" (Föppl 1891a, 15). Föppl's views on the bridge collapse was only reported in passing in the discussion on the causes of the failure, so that he decided to write a manuscript on spatial framework theory. He wrote this manuscript between 21 June 1891 and 22 September 1891, although due to a printers' strike it was not published until early 1892 under the title *Das Fachwerk im Raume* (spatial frameworks) (Fig. 9).

Having made a distinction between free frameworks (*"Fachwerk"*) and bound frameworks in the form of trussed girders (*"Fachwerkträger"*), he developed mathematical stability criteria for free spatial frameworks. In 1880 he had already formulated criteria for planar frameworks (Föppl 1880, 7–11) and provided proof in 1887 (Föppl 1887). Föppl's condition of a free spatial framework with s members and k joints, i. e.

$$s \geq 3k - 6 \tag{2}$$

is necessary for stability or rigidity, but not sufficient (figure 11). According to Föppl a free spatial framework is only stable, i.e. rigid or not kinematic, if the Jacobian functional determinant of order $s = 3k - 6$ for the implicit functions of the distance squares of joints $f_r(x_i, x_k, y_i, y_k, z_i, z_k)$, which are connected through members, is not zero (Föppl 1892, 6–11). This sufficient criterion for stability of free spatial frameworks is "rather useless in practice", as Föppl commented (Föppl 1892, 9), although he used it as a basis for deriving a very important theorem on statically determinate trussed frameworks, for which inequality (2) is transformed to equation

$$s = 3k - 6 \tag{3}$$



s = number of members
$s = 6$

k = number of joints
$k = 4$

Necessary stability criterion:
$s \geq 3 \cdot k - 6$
$6 \geq 3 \cdot 4 - 6 = 6$

$$f_1 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 - l_{12} = 0 = f_1\ (x_1, x_2, y_1, y_2, z_1, z_2)$$
$$\vdots$$
$$f_r = (x_i - x_k)^2 + (y_i - y_k)^2 + (z_i - z_k)^2 - l_{ik} = 0 = f_r\ (x_i, x_k, y_i, y_k, z_i, z_k)$$
$$\vdots$$
$$f_6 = (x_3 - x_4)^2 + (y_3 - y_4)^2 + (z_3 - z_4)^2 - l_{34} = 0 = f_6\ (x_3, x_4, y_3, y_4, z_3, z_4)$$

Jacobian functional determinant of order $3k - 6 = 6$:

$$\Delta = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial z_2} \\ \vdots & & & & & \\ \frac{\partial f_r}{\partial x_i} & \frac{\partial f_r}{\partial x_k} & \frac{\partial f_r}{\partial y_i} & \frac{\partial f_r}{\partial y_k} & \frac{\partial f_r}{\partial z_i} & \frac{\partial f_r}{\partial z_k} \\ \vdots & & & & & \\ \frac{\partial f_6}{\partial x_3} & \frac{\partial f_6}{\partial x_4} & \frac{\partial f_6}{\partial y_3} & \frac{\partial f_6}{\partial y_4} & \frac{\partial f_6}{\partial z_3} & \frac{\partial f_6}{\partial z_4} \end{vmatrix}$$

Sufficient stability criterion:

$\Delta \neq 0$ ⟶ stable (not kinematic)
Example: Tetrahedron

$\Delta = 0$ ⟶ unstable (kinematic)
Example: If P_4 falls within the triangular plane formed by $P_1P_2P_3$ the necessary stability criterion is met, but not the sufficient criterion.

Figure 11
Föppl's stability criterion illustrated using the example of a free tetrahedron framework

"A stable trussed framework only containing the necessary number of members (i. e. meeting Eq. (3) —author's note) is also statically determinate and vice versa, i. e. it is stable if it is statically determinate for all loads" (Föppl 1892, 30). Föppl used this theorem as a basis for a coherent theory of statically determinate spatial frameworks. He went on to formulate an outline for the theory of statically undetermined spatial frameworks.

In the second part of his book, without using formulas Föppl composed numerous structural forms of spatial frameworks such as trussed shells, network domes, spatial bridge systems and spatially braced truss systems. He also developed a new approach for analysing familiar systems such as the Schwedler dome, thus systematically making the third dimension accessibly to structural theory. His work is a rare example of the great heuristic potential of theoretical thinking in structural theory, the horizon of which we only reached not all that long ago. Nevertheless, it was not realised in structural theory. From a historical/logical point of view, Föppl got as far as the implicit mathematical structural law of Plato, and merged it with his static law and formation law into the composition law for spatial frameworks. In Föppl's work grace and law thus appear in an aesthetic adoption of possible artistic forms of spatial frameworks by means of mathematical-physical cognition of their composition law.

Integration of spatial framework theory into classic structural theory

As a consulting engineer Müller-Breslau undertook a structural analysis of the Zimmermann dome above the plenary chamber of the Reichstag in Berlin, which had been completed in 1890. Two years earlier he had succeeded Emil Winkler (1835–1888) in the chair for structural design and bridge construction at TH Berlin. In this capacity, Müller-Breslau consolidated his structural techniques, which he had summarised in the form of monographs in 1886 (Müller-Breslau 1886) and 1887 (Müller-Breslau 1887) into a theory of linear-elastic frames. He thus laid the keystone for the discipline formation period of structural theory between 1825 and 1900. The development of spatial framework theory by Müller-Breslau in 1891/92, using the methodology of planar framework theory, represented a moment of completion of this development period towards classic structural theory (Müller-Breslau 1891 and 1892). It is significantly differeret from Föppl's work (Föppl 1892), since Müller-Breslau moved within the disciplinary framework of classic structural theory, for which he developed a symbolic language. Müller-Breslau did not work deductively, he used examples to develop his spatial framework theory, i. e. he worked inductively. The inductive method continued to shape structural theory until well into the second half of the 20th century.

Having explained spatial dynamics using statically determinate domes and a bridge girder as examples, he went on to suggest a general technique for calculating

the member forces of statically determinate spatial frameworks, i. e. his member replacement method (Müller-Breslau 1891, 439–440). The member replacement method even leads to success in cases where forces in joints with three to six members cannot be analysed directly. “By removing members and adding the same number of new members, referred to as replacement members, the trussed framework can be transformed into a simpler structure, if possible a structure with tension forces that can be determined by repeatedly solving the task of decomposing a given force in three directions” (Müller-Breslau 1891, 439).

Müller-Breslau’s member replacement method, which he had already alluded to in 1887 through two examples (Müller-Breslau 1887, 207–208 and 213–214), structurally corresponds to his force method for analysing statically undetermined systems: the member replacement method and the force method are based on equivalent formulations (Fig. 12). The member replacement method can be used to calculate the member forces of complex statically determinate systems using clearer substitute systems, and to verify the stability of the initial system. For the latter Müller-Breslau provided a sufficient stability criterion (Müller-Breslau 1891, 439) that can be verified directly and much more easily than with Föppl’s technique. This demonstrates the superiority of structural techniques based on linear algebra compared with the pure mathematical technique in the form of the Jacobian functional determinant. Müller-Breslau’s member replacement method is particularly suitable for analysing complicated statically determinate spatial frameworks. He also drew attention to the technique Henneberg (1886) suggested in 1886 (Müller-Breslau 1891, 440), although Henneberg’s technique was limited to free trussed frameworks, whereas Müller-Breslau’s member replacement method is also suitable for bound systems. Henneberg had provided the methodological basis for Müller-Breslau’s member replacement method, but remained within the realm of the mechanics of rigid bodies, since Henneberg was more interested in reducing complex framework systems to simpler systems, i. e. in the mathematical-physical derivation of the formation law for trussed frameworks. Müller-Breslau’s member replacement method, on the other hand, was aimed at clear and efficient structural analysis of complex trussed frameworks, although his contribution to spatial framework theory went beyond the member replacement method. Having analysed the Schwedler dome, he extended the kinematic theory of statically determinate planar trussed frameworks to spatial frameworks. Using the structural calculation of a regular, octagonal truncated pyramid, statically indeterminate to the second degree, as an example, Müller-Breslau finally convinced readers of the practicability of the force method for analysing statically undetermined spatial frameworks. He thus succeeded in calculating spatial frameworks using classic structural theory techniques that had been developed for linear-elastic frames.

Force method	Member replacement method
n-fold statically indeterminate system (initial system)	Statically determinate system (initial system)
Transformation into a statically determinate basic system through release of n ties	Transformation into a statically determinate substitute system through replacement of n members
Calculation of the translation steps δ_{i0} and δ_{ik} in the statically determinate basic system: – δ_{i0} : Translational step at point i due to initial state (given load) – δ_{ik} : Translational step at point i due to the force parameter $X_k = 1$	Calculation of the member forces S_{i0} and S_{ij} in the substitute system: – S_{i0}: Member force at point i in the member inserted at this point due to the original state (given load) – S_{ij} : Member force at point i in the member inserted at this point due to the member force $Z_j = 1$ resulting from the removal of the member at point j
Compliance with the n elasticity conditions of the statically indeterminate system (continuity statements for deformation parameters): $[\delta_{ik}] \cdot [X_k] + [\delta_{i0}] = [0]$	Compliance with the n equilibrium conditions of the statically determinate initial system (continuity statements for force parameters): $[S_{ij}] \cdot [Z_j] + [S_{i0}] = [0]$
Calculation of the n released statically indeterminates X_k of the initial system via n elasticity equations	Calculation of the n removed member forces Z_j of the initial system via n equilibrium conditions
	Sufficient stability criterion: $\det\ [S_{ij}] \neq 0 \longrightarrow$ initial system is not kinematic $\det\ [S_{ij}] = 0 \longrightarrow$ initial system is kinematic

Figure 12
Form equivalence of the member replacement method and force method according to Müller-Breslau

During the course of a series of essays by Müller-Breslau on the calculation of spatial frameworks in 1902 and 1903, a dispute developed with Otto Mohr regarding techniques for calculating spatial frameworks (Kurrer 2004, 613) that became rather polemic.

Henneberg incorporated spatial framework theory into the system of theories dealing with the mechanics of rigid bodies as part of graphical statics based on mathematics (Henneberg 1894, 1902, 1903 and 1911). Graphical statics reached its practical limits with the analysis of spatial frameworks and statically indeterminate systems. His disciple Schlink took up Föppl's trussed shell idea and developed multiple trussed shells enclosing one or several voids (Schlink 1907). The second edition of his work *Technische Statik. Ein Lehrbuch zur Einführung in das technische Denken* (Technical statics. An introducing into technical thinking) appeared in 1946, where Schlink described the corresponding planar projection technique developed by Mayor (1910 and 1925), v. Mises (1917), Prager (1926 and 1927) and Sauer (1940), which reduces spatial force problems to planar ones (Schlink and Dietz 1946, 305–314).

This level of cognition of spatial framework theory remained unsurpassed until the second half of the last century. Practical applications included structural calculations for dome, crane and tower constructions, construction of conveyor bridges for waste material, as well as bridge and aircraft construction.

During its consolidation period between 1900 and 1950, structural theory thus lost the unity of static law, formation law and structural law in the shape of the composition law for spatial frameworks that had been alluded to by Föppl in 1892, because the primary focus was on the static law.

Spatial frameworks in an era of technical reproducibility

Schwedler domes (Fig. 13 a), network domes (Fig. 13 b), Zimmermann domes (Fig. 13 c) and Schlink domes (Fig. 13 d) consisted of rolled steel profiles of different length that were riveted in the joints. Despite the fact that industrially produced steel profiles were used, each dome was unique. Only the introduction of welding and the typification of joints and members paved the way for mass production of spatial frameworks. Nevertheless, a conflict between production and assembly of components on the one hand and structural calculations on the other hand began to emerge during the period of high industrialisation between 1890 and 1914 that entered contemporary awareness during the inter-war years: While the scientific development of structural theory tended towards a procedural, syntactic and operative use of symbols, thereby creating a special intellectual technology, practical structural calculations continued to use a hands-on approach, although they too were affected by the rationalisation movement that began to

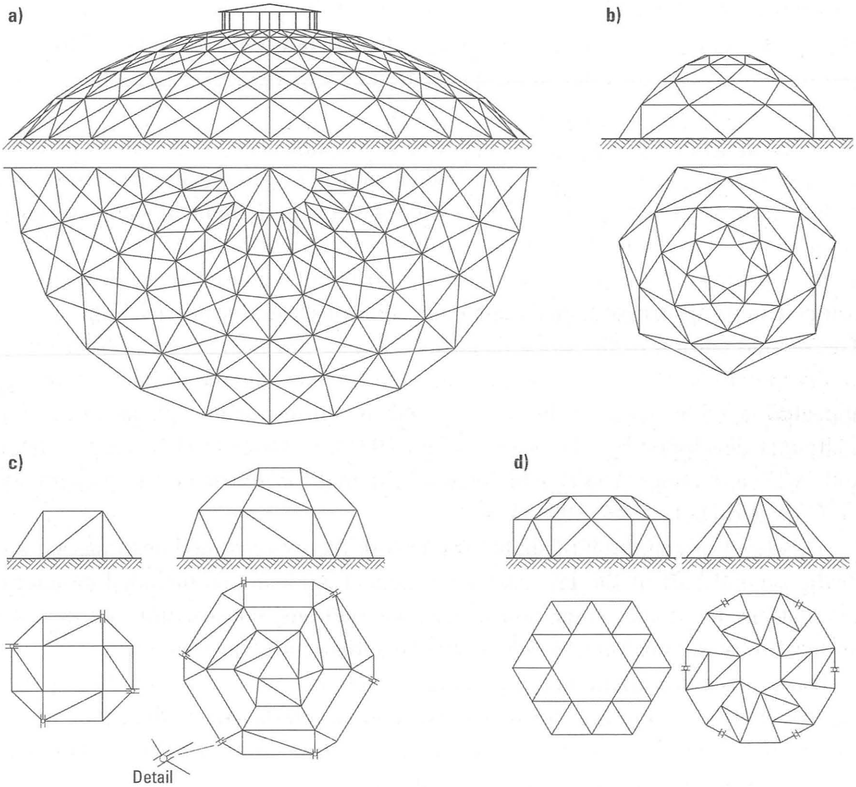


Figure 13
Dome types; a) Schwedler dome, b) Network dome, c) Zimmermann dome, d) Schlink dome

spread after the 1st World War. As part of this rationalisation movement structural engineer Konrad Zuse (1910–1995) started to automate structural calculations. Between 1936 and 1941 he developed the first functioning computer, although computers only started to be used for the calculation spatial frameworks towards the end of the 1960s. On the other hand, inventors such as Alexander Graham Bell (1847–1922), design engineers such as Wladimir Grigorjewitsch Schuchow (1853–1939), autodidacts such as Richard Buckminster Fuller (1895–1983), and inventors and entrepreneurial engineers such as Max Mengeringhausen (1903–1988) had already taken the first steps towards mass production of spatial frameworks decades ago.

Alexander Graham Bell

The first prefabricated spatial framework was designed by Alexander Graham Bell in early 1907 (Fig. 14). It consisted of relatively small tetrahedrons. The inventor of the telephone was also engaged in research into aircrafts and shipbuilding, medicine, electrical engineering, biology and engineering sciences. For example, Bell built a glider model consisting of a spatial framework covered with canvas. Later he developed gliders that were capable of carrying people. Bell standardised member

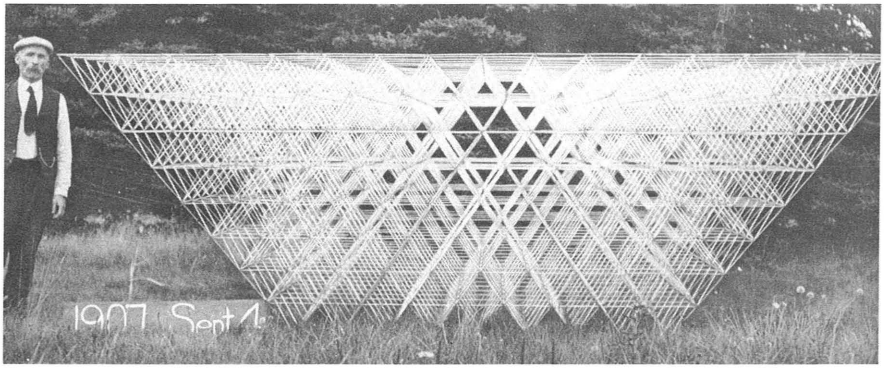


Figure 14
First prefabricated spatial tetrahedron framework by Bell (Makowski 1963, 36)

and joint elements, commissioned the production of standardised tetrahedrons (Wachsmann 1989, 31), and assembled them to form complex spatial frameworks according to the first formation law. In Bell's assembly technique, formation law and structural law for spatial frameworks form a materialised technological unit determined by function, design, natural form and artistic form. Procedures on construction sites thus started to tend towards standardised forms and developed into the scientific subject of construction management that started to emerge during the comprehensive rationalisation movement of the 1920s.

Wladimir Grigorjewitsch Schuchow

"Engineering is thankless", Schuchow told his grandson, "because you have to possess knowledge in order to understand its beauty" (F. V. Schuchow 1990, 21). Schuchow invented, designed, calculated and built spatial structures of breathtaking beauty. In 1894 he applied for a patent for his meshed roofs (V. G. Schuchow 1990, 175). These tensioned structures consist of meshed steel strips and angle

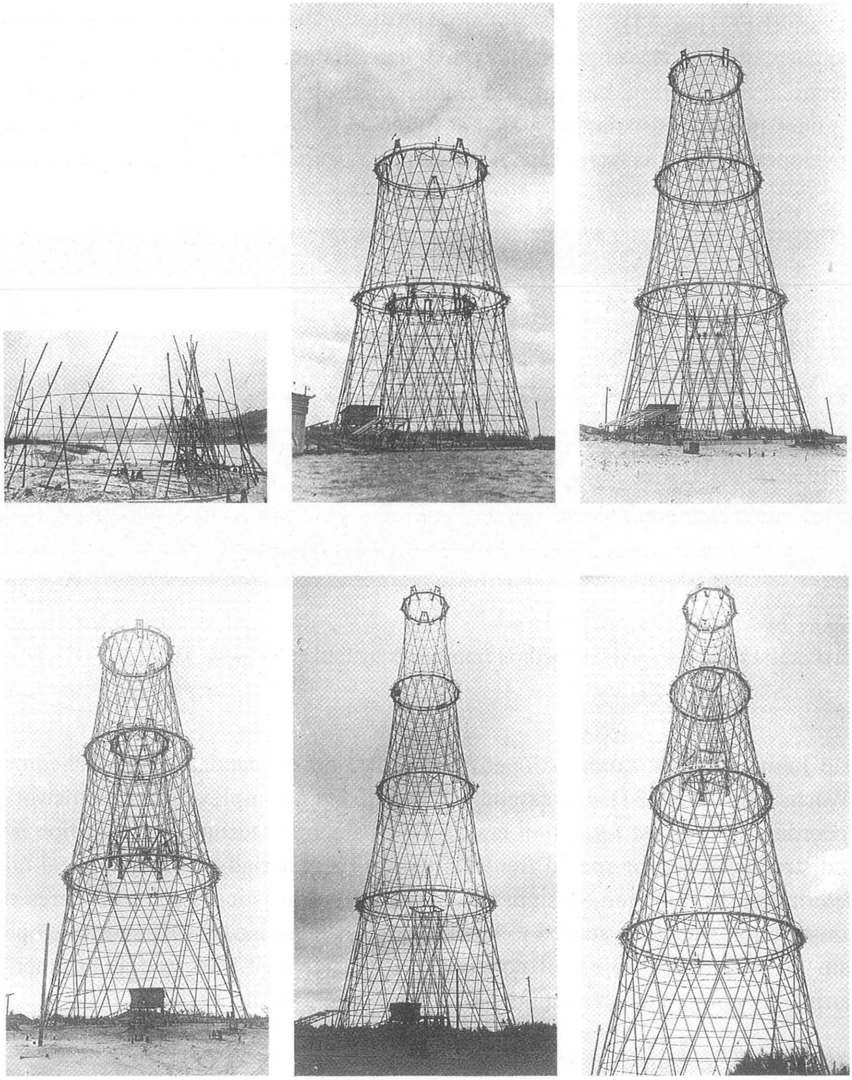


Figure 15
Production and assembly of the 120 m high, five-storey NIGRÉS electricity mast by Schuchow on the river Oka, 1927–1929 (Petropavlovskaja 1990, 99)

sections. Schuchow's trussed shells are similar to those developed by Föppl, although Schuchow moved away from the trussed framework principle. Meshed roofs and trussed shells are made from identical parts and riveted or screwed together in the joints. In his patent applications Schuchow described the benefits of his spatial structural systems (Graefe 1990, 28):

- Substantial weight reduction compared with ordinary roof structures;
- The components of meshed roofs are only subject to tension, those in trussed shells only to pressure;
- High load-carrying capacity of the meshed surfaces, even for concentrated loads;
- Simplified production and assembly through uniform structural elements.

As chief engineer of the Moscow-based company Bari Schuchow used his new roof types for eight halls at the 1896 the pan-Russian exhibition at Nishni Nowgorod, where he also showed a tower structure in shape a hyperboloid. This structural system became widespread for the construction of water towers, radio towers and electricity pylons. Figure 15 illustrates the telescopic assembly principle for a such tower structure in the shape of the 120 m high, five-level NIGRÉS electricity pylon at the river Oka: The hyperbolic sections were assembled inside the structure and raised using five wooden crane frames.

Schuchow's cost-effective and graceful structures accompanied the industrialisation of Russia right until the time of the first Soviet five-year plan.

Richard Buckminster Fuller

In 1954 Richard Buckminster Fuller was awarded a US patent for the construction of geodetic domes. The principle was based on the projection of an icosahedron inscribed a sphere onto the surface of the sphere (Fig. 16). The result are 20 equilateral spherical triangles, each of which can be subdivided into six further triangles through their three apothems, which lie on great circles. The frequency of the first subdivision is $\nu = 2$, since the side of the basic triangle is split into two identical sections. For large-span domes further subdivisions with $\nu = 4$, $\nu = 8$, $\nu = 16$ etc. are required, because otherwise the slenderness ratio of the members would become excessive. Nevertheless, even for higher frequencies the number of different member lengths is relatively small. While the surface of the sphere can be divided into a maximum of 20 equilateral spherical triangles, complete division into hexagons is no longer possible, and at least 12 pentagons have to be used. This basic fact of sphere geometry is illustrated by a football as a true image of a Fuller dome.

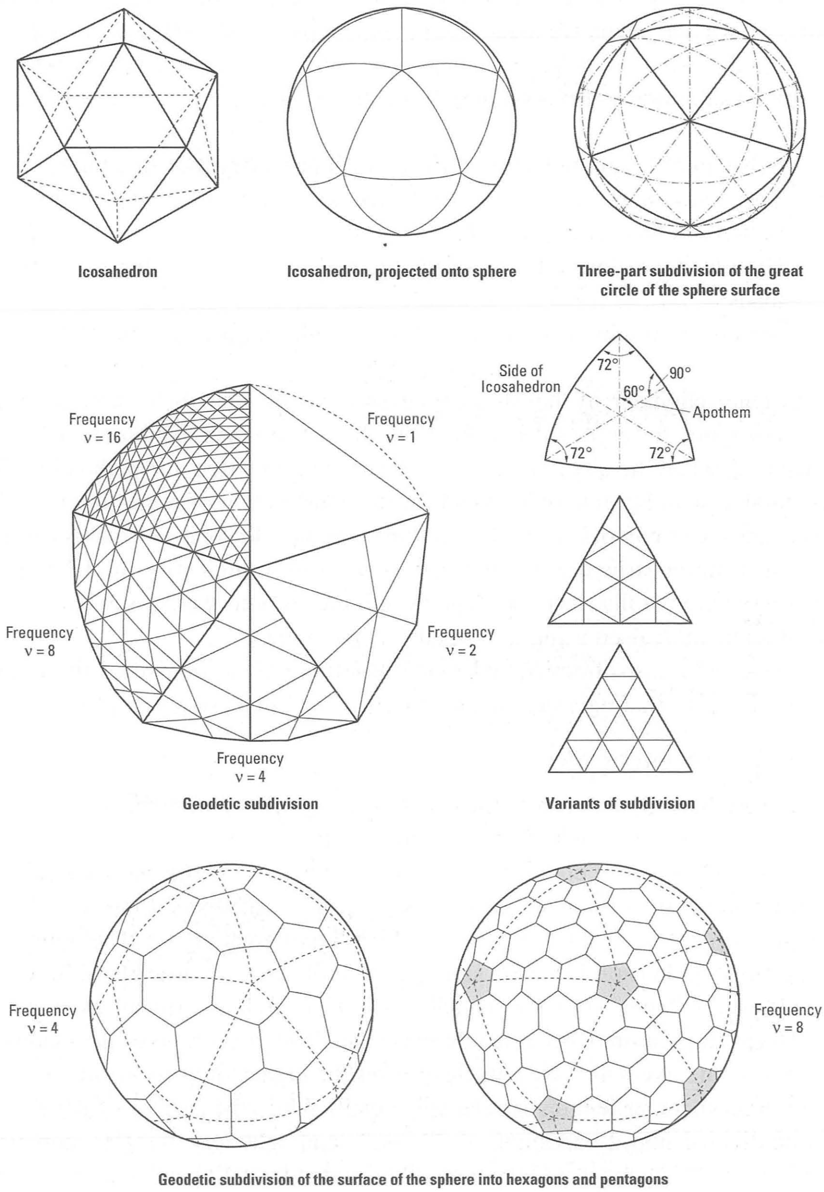


Figure 16
Geodetic domes after Richard Buckminster Fuller

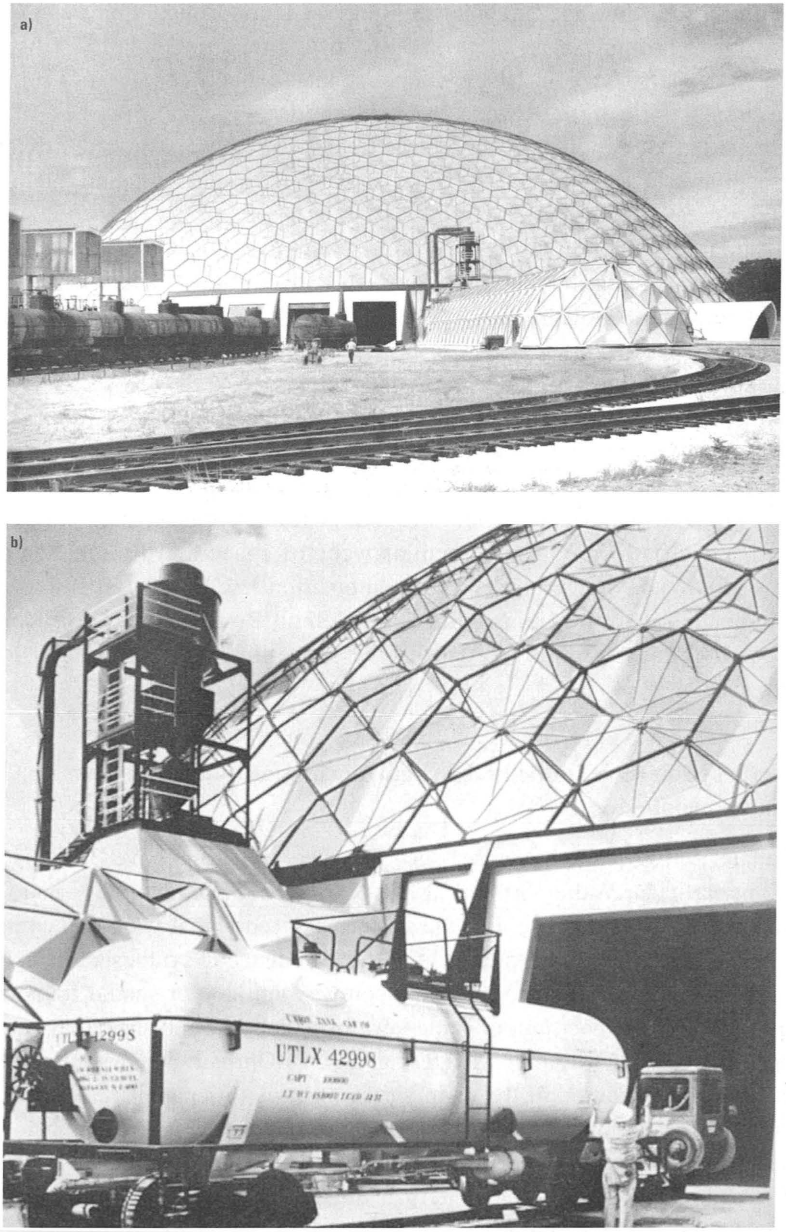


Figure 17
a) Dome at Baton Rouge, Louisiana, constructed in 1959 (Makowski 1963, 133)
b) Structural detail (Makowski 1963, 134)

Figure 17 (a) shows the geodetic steel dome at Baton Rouge, Louisiana with a span of 115 m, completed in 1959. The two-layer structure with a layer distance of 1.20 m is characterised by the frictional connection between the roof skin and the hexagonal skeleton framework, forming a folded structure member grid. The dome consists of 321 hexagonal welded steel panels, to which the external hexagonal layer consisting of steel tubes is connected via diagonal ties and vertical corner members, figure 17 (b).

In the late 1950s Fuller's geodetic domes were used for housing radar equipment (radomes) along the 5,000 km long US early warning system between the Arctic Circle and 60 degrees north (Marks 1960, 462). The highlight in the construction of geodetic domes was the three-quarter dome with a diameter of 76 m of the US pavilion at EXPO 1967 in Montreal.

With Fuller's geodetic dome system non-Euclidean geometry in the shape of sphere geometries started to be used for the construction of spatial frameworks. At this level he engaged in research on the interaction between the formation law and the structural law for spatial frameworks. In this way Fuller created industrially produced "peculiar webs of space and time" (Benjamin 1989, 355). With Mengeringhausen's composition law for spatial frameworks and associated work by Helmut Eberlein, Helmut Emde and Herbert Klimke, the spatial framework was freed from the space-time that was bound to the sphere.

Dialectic synthesis of individual structural composition and mass production

In the late 1950s Mengeringhausen succeed where the progressive Bauhaus masters —in particular Walter Gropius (1883–1969), whom he admired— had failed in steel construction practice, i. e. the synthesis of individual structural composition and mass production. A precondition for this dialectic synthesis was his cognition and systematic application of the composition law for spatial frameworks in the form of the higher unit of static law, formation law and structural law. The MERO technique (MERO = "Mengeringhausen ROhrbauweise") represents the practical implementation of the eight structural laws for spatial frameworks formulated by Mengeringhausen in 1940 (figure 18).

The breakthrough for Mengeringhausen's MERO framework for large-span roof structures came in 1957 at Interbau in Berlin, where in collaboration with architect Professor Karl Otto he created a spatial framework grid consisting of semi-octahedrons and semi-tetrahedrons (Fig. 19). The rectangular spatial framework covered an area of 52 × 100 m it was exclusively constructed from standard MERO joints and a single type of standard members with a system dimension of

I Regular spatial frameworks are composed from equilateral and/or rectangular triangles such that platonic bodies or shapes derived from them are created.	**II** Due to their regular structure spatial frameworks are statically ideal. Uniform joints and members of similar length enable industrial series production.
III The length of the members of a spatial framework form a geometric series with a natural growth of $\sqrt{2}$.	**IV** Series of similar polyhedrons can be built using n different member lengths from the geometric series of natural growth.
V For the similar polyhedrons the sizes of the surfaces form a geometric series with factor 2, and the volumes form a geometric series with factor $2 \cdot \sqrt{2}$.	**VI** All elementary bodies mentioned above, their derivatives and the associated composite spatial frameworks can be formed using a single universal joint and members from the geometric series of natural growth.
VII The universal joint is a body with 26 surfaces from the series of semi-regular Archimedean bodies whose 18 square have the same distance to the centre of the body and feature concentric holes.	**VIII** Regular spatial frameworks can be constructed from a universal joint with 18 joints in the form of the standard MERO joint.

Figure 18
Eight structural laws of Mengeringhausen (Mengeringhausen 1983, 114–115)

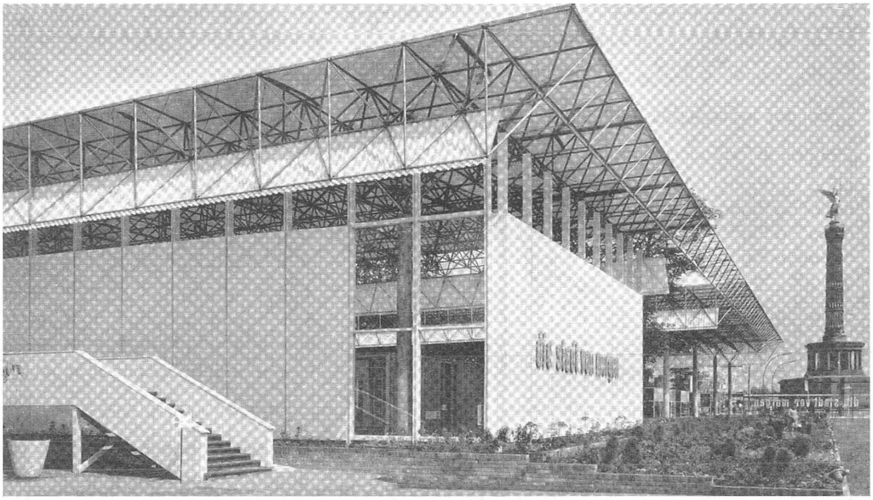


Figure 19
Hall of the town of the future at Interbau Berlin 1957 with visible spatial framework as a design element (Mengeringhausen 1975, 129)

2 m, plus screws with M20 threads. The structural law corresponds to hexagonal close sphere packing. The tetrahedrons are placed from above onto the upright semi-octahedrons, creating a rectangular framework (Fig. 20). Such plate-like spatial frameworks are statically indeterminate to a high degree and could not be calculated using member statics. In the early 1970s engineers therefore had to make do with the differential technique applied to plate theory (Lederer 1972). The basic principle of this finitisation of the spatial elastic continuum through the spatial framework can already be found at Kirsch (Kirsch 1868), Hrennikoff (Hrennikoff 1941). It contributed to the development of the finite element method in the 1950s. For structural analysis of spatial frameworks the finite element method only became significant in the 1970s. The relationship between static law and structural law of spatial frameworks thus remained superficial for the time being.

In 1962 Mengeringhausen published a brochure entitled *Komposition im raum* (spatial composition) (Mengeringhausen 1962) on the occasion of *Debau Essen*, in which he made an attempt to systematically relate his structural laws, the formation law and the static law of spatial frameworks to each other. However, it was not until 1966 that Mengeringhausen presented his *Kompositionslehre räumlicher Stab-Fachwerke* (composition theory for spatial member frameworks) (Mengeringhausen 1967) at the *International Conference on Space Structures* in

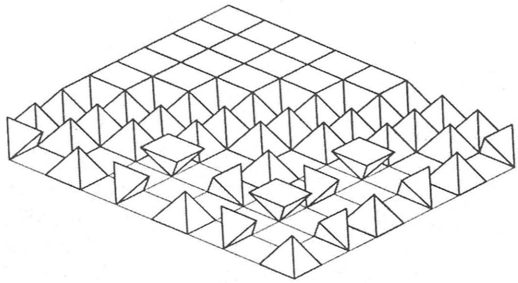


Figure 20
Composition of a slab-shaped, rectangular spatial framework consisting of semi-octahedrons (1/2O) and tetrahedrons (T): 1/2O + T (Mengeringhausen 1983, 72)

London. He coded the basic bodies of spatial frameworks in analogy to the symbolic language of crystallography and chemistry (Fig. 21), in order to classify spatial frameworks composed of such bodies using structural formulae (Fig. 20). The generalisation of his structural laws to form the composition law for spatial frameworks was based on the *discourses on symmetric polyhedrons* published in 1849 by Auguste Bravais (Bravais 1892), the co-founder of crystallography. Fritz Kesselring's book *Technische Kompositionslehre* (technical composition theory), published in 1954 (Kesselring 1954), may well also have provided inspiration for Mengeringhausen. In his doctoral dissertation on geometric derivation of the two-layer frame grids from cubical grids (completed in 1970), Helmut Eberlein significantly deepened and expanded Mengeringhausen's classification system for spatial frameworks using crystallography-based terminology (Eberlein 1970).

Towards the end of the 1960s, the use of computers led to a revolution in both the theoretical and the practical composition of spatial frameworks. It started with the construction of the German concert dome at EXPO 1970 in Osaka (Fig. 22) and was completed in 1979 with the construction of the shell-type roof for the grandstand of the sports stadium in the Dalmatian coastal town of Split with a span of 200 m (Fig. 23). It would have been impossible to manually calculate 839 different joints with a total of 3,460 joints and 1143 different members with a total of 12,382 members for the grandstand roof at Split. The craftsmen of structural calculations who left impressive, intellectually created artistic forms, developed into *Geistwerker* (a term created by Mengeringhausen, meaning "mind workers") such as Helmut Emde and Herbert Klimke, who pushed ahead with the systemic integration of design, calculation, construction and production of derived spatial frameworks through computers.

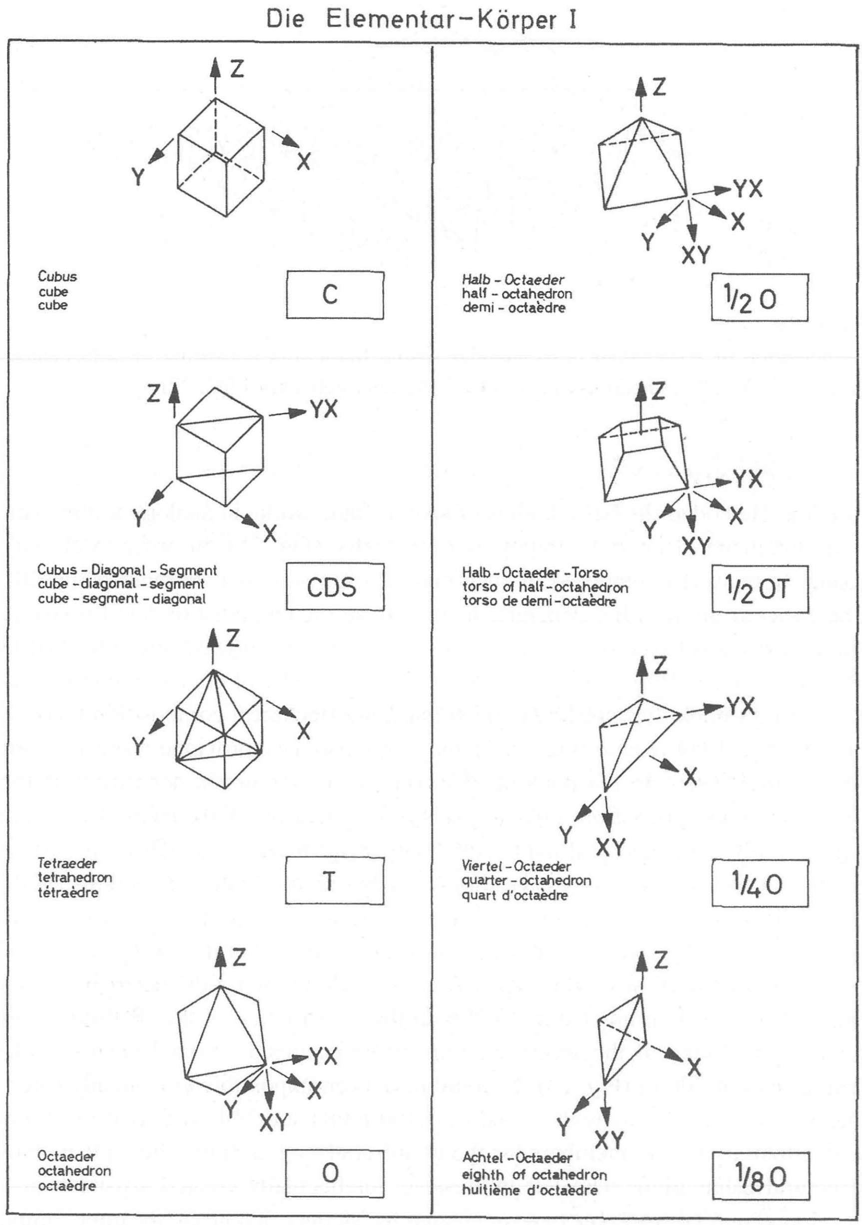


Figure 21
Elementary body according to Mengeringhausen for the classification of regular spatial frameworks (Mengeringhausen 1975, 45)

Figure 22
Concert dome of the Federal Republic of Germany at EXPO 1970 in Osaka (Mengeringhausen 1975, 293)

Figure 23
Stadium grandstand roof at the Dalmatian coastal town of Split (Kurrer 2004, 621)

Helmut Emde expanded the geometry from planar to curved spatial frameworks, generated computer-aided framework topologies, thereby recording derived spatial framework geometries by computer (Emde 1977). Having become director of the MERO computer centre on 1 January 1974, Herbert Klimke succeeded in breaking the linear dominance in the structural analysis of spatial frameworks by utilising the insights gained during his doctoral dissertation and be modifying the *Structural analysis program* (SAP) finite element software system in such a way that nonlinear 2nd order theory effects and ideal-elastic, ideal-ductile material behaviour could be considered (Klimke 1976). For non-standardised grids, Jaime Sanchez (1980), Martin Ruh and Herbert Klimke (1981) described the topological relationships of spatial frameworks in an integer grid and the metric through a coordinate transformation (Klimke 1983, 258–259), thereby consistently continuing Helmut Emde's work. With the introduction of NC-controlled production of members and joints and automatic generation of location plans etc., the company MERO completed the systemic link between design, calculation, construction and production of derived spatial frameworks through the computer and took on the role of pacemaker in the construction industry, particularly for steel construction. Mengeringhausen thus completed the dialectic synthesis of individual structural composition and mass production of spatial frameworks. The composition law for spatial frameworks, comprehensively described in his books (Mengeringhausen 1975 and 1983) (Fig. 2 b) can be interpreted as follows:

- Due to the large number of members s and joints k, the formation law for spatial frameworks ($s = 3k - 6$) can no longer be calculated manually;
- In computer statics the formation law for spatial frameworks is represented as a pure mathematical transformation;
- With computer-aided design the formation law merges with structural law, so that the difference between geometry and statics disappears;
- The static law no longer appears in the form of force diagrams or elasticity conditions, but becomes an inextricable component of the trinity of material, translation and equilibrium law for finite member elements; and finally;
- The formation law, structural law and static law of spatial frameworks only developed into the higher unit of the composition law via the linkage of design, calculation, construction and production through the computer as a symbolic machine.

The composition law thus not only materialises itself as an option of grace in the finished spatial framework, but also carries it in the formation process leading to it, which is enlivened through human activity. In this formation process as-

pects that stimulate and satisfy desire, grace or sensory beauty can also be discovered.

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Boscovich's contribution to the theory of domes: the expertise for the Tiburio of Milan cathedral

Gema López Manzanares

In 1763, Boscovich, one of the three authors of the famous *Parere di tre mattematici sopra i danni della cupola di San Pietro*, was in charge of studying the stability of the oval dome of the Caesarea Library in Vienna. This dome showed alarming cracks and although he did not include numerical calculations in his expertise, he made a detail study of the movements suffered by the counterparts and proposed to encircle the dome with iron rings. In this expertise he demonstrated himself as very knowledgeable about this type of constructions (López 2005).

Two years later, Boscovich must face up to a new problem: the construction of the spire over the dome of the Tiburio of Milan cathedral. The problem was different here than in the other expertises, so that the scholars had to predict the future behaviour of the dome charged with the weight of the spire. Once again, Boscovich made an analysis of great interest, where he included numerical calculations resulting from applying the Virtual Work principle to a collapse mechanism of the dome caused by sliding down of the keystone with the spire. Besides, he stated in brightness way his theory about vaults and domes, and he got ahead of Coulomb in the calculation of the worst location of the collapse hinges. Also, P. Regi was asked for his opinion, who applied La Hire's method in Bélidor's version to study the stability of the dome and the piers.

These contributions, in an intermediate period between the analysis of the stability of Saint Peter's dome and the construction of Sainte Genevieve's church or Pantheon of Paris, that also will be object of a great controversy about the stability of its piers, are of great importance to understand the development of the scientific theory of the domes that occurs along XVIII century.

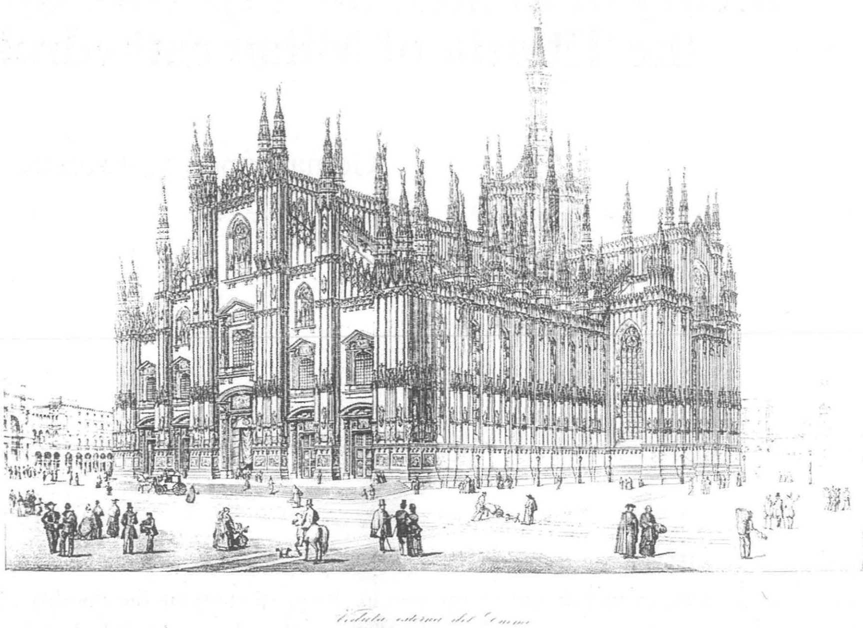

Figure 1
Elevation of the Duomo of Milan (Anonymous, s.a.)

The construction of the spire of the Tiburio in Milan

In 1762 the chapter of the Duomo of Milan decided to build a great spire that would crown the dome or Tiburio over the transept, projected and built by Amadeo at the beginning of the XVIth century. This spire was already envisaged in the initial projects but the construction of the dome stopped in 1640 and since then, there was no talk of it (Nava 1845, 11; Scaglia 1982).

The architect Francesco Croce was in charge of studying those projects and preparing the construction of the spire. Two years later, on 25th May 1764, Croce showed a wooden and wax model and a written plan about his project, but before its approval the project was submitted to the judgement of several experts. Between them there were two mathematicians, Boscovich and Regi, who had to analyse the effect of the weight of the spire over the stability of the building. The other experts were an anonymous one, Beccaria and Francesco Martinez, who signed off the last of these expertises on 13th May 1765. On 8th July the model proposed by Croce was approved and the spire was built in less than four years (Nava 1845; Benvenuto 1991, 371–374; López 1998, 522–556; Repishti 2003; Stolfi 2003).

Description of the dome

The dome or Tiburio rises over the transept of the Duomo of Milan (Fig. 2). Four main levels can be distinguished (Boscovich 1765, 56–8; Regi 1765, 69):

1° The first one corresponds to the piers and main arches until the springing of the dome itself. The main arches are pointed, with a radius equal to the span between the inner edges of the piers. Over them there are other incomplete arches that receive the weight of the eight ribs of the dome and concentrate it towards the piers; the octagonal plan is solved in the angles by a kind of rib pendentive.

The pier base is equivalent to a circle with a diameter of 5 arms (2.97 m) and their height until the springing of the main arches is greater than 51 arms (30.34 m). They form a square in plan of 32 arms side (19.04 m). Regi states that the piers are forty arms height in the minor naves and fifty three in the main nave.[1]

2° The dome rises over the described octagonal plan, at a height of one third of the span over the main arches keystone. Its eight ribs concentrate the weight towards these arches. The vertical drum rises outside, with 23 arms height (13.68 m) and 2 arms width (1.19 m), with a brick core and a marble revetment, and counterforts in the angles that increase its width until three arms (1.78 m). The

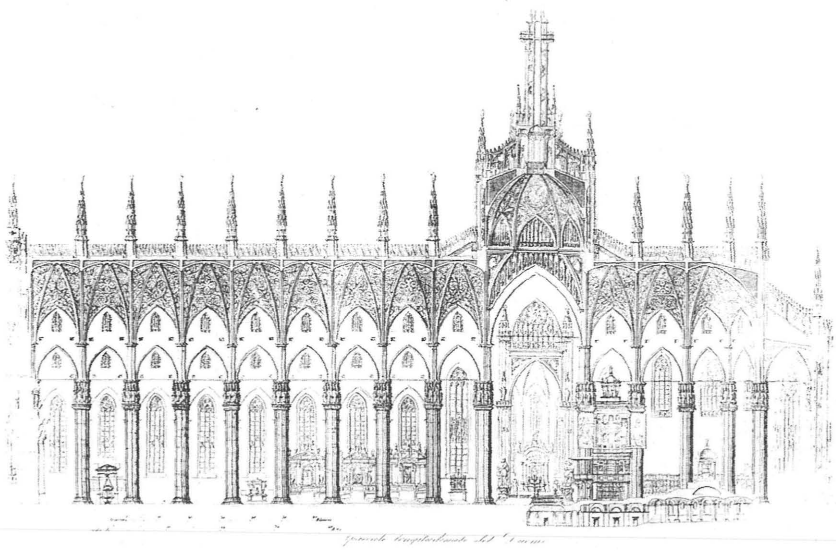

Figure 2
Longitudinal section of the Duomo of Milan (Anonymous, s.a.)

springing of the dome is located at the base of the drum, not at the upper edge; several transverse walls with passing openings join the dome and the drum. The dome is embedded in the drum until a height of 2 fi arms (1.5 m). These ribs are of marble and the joints between the great blocks of stone converge towards the corresponding centre of curvature. Between these ribs it was built an inner pointed shell made of brick and with 9 ounces width (23 cm); the extrados shell is nearly horizontal. The inner shell shows unloading pointed arches over the windows, in such way that it transmits its weight toward them and the ribs that concentrate it over the piers. In the upper part the ribs and the shells converge in a marble ring of great width that serves as a base for the lantern. In the angles, over the eight marble counterforts of the drum, there were to be built eight small spires and lattice masonry over the ribs that would join them to the lantern.

There were a lot of iron ties in the naves, but also two iron octagonal rings crossing through the windows of the dome: the one at the bottom part at a height

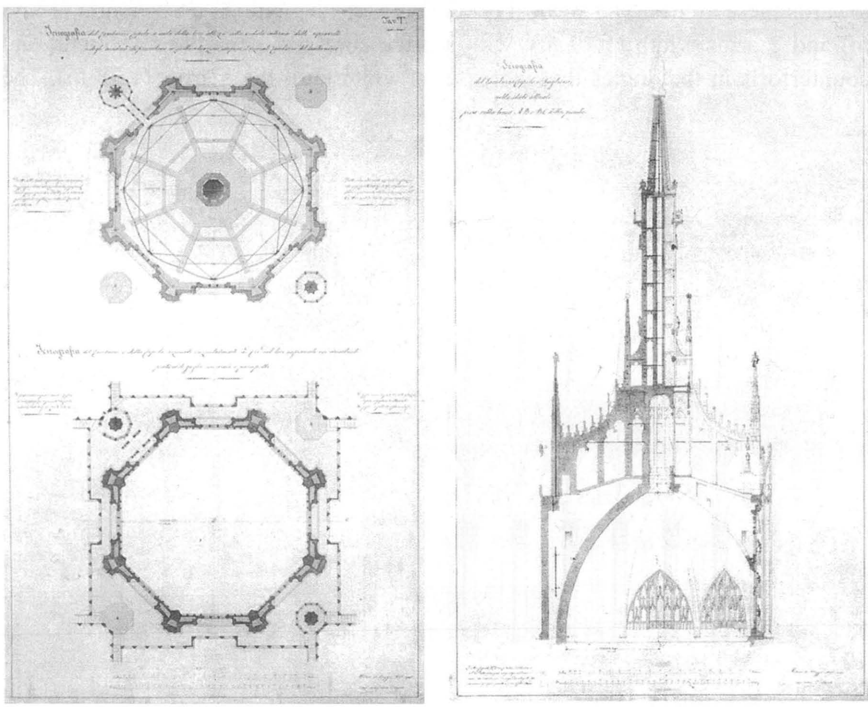

Figure 3
Plans and sections of the Tiburio by Pietro Pestagalli 1843 (Stolfi 2003)

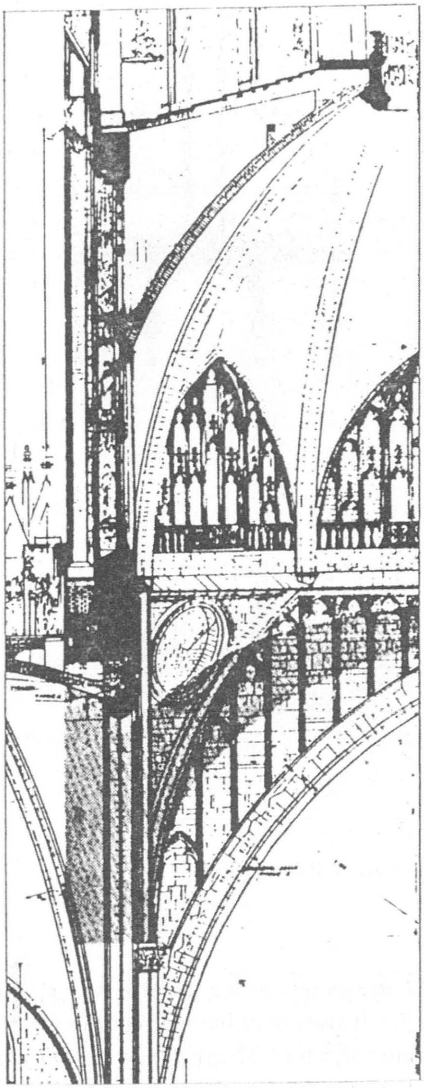

Figure 4
Detailed section of the Tiburio of Milan (Scaglia 1982)

of 8 1/2 arms (5.00 m) and the upper one, at a height of 14 (8.30 m). Their transversal section had 11 × 18 puntos^2 (4.5 × 7.5 cm^2). Besides there were a third hidden iron ring over the windows at a height of more than 30 arms (18 m).[2]

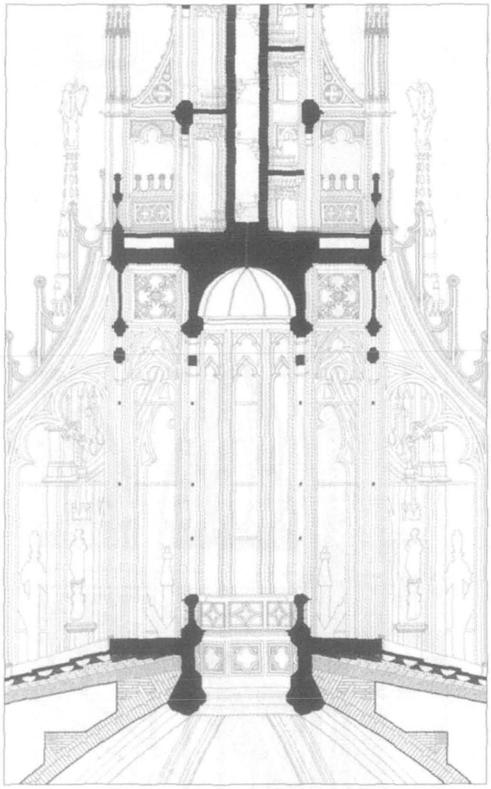

Figure 5
Section of the lantern (Ferrari da Passano 2003)

3° The third level corresponds to the lantern, which was not concluded, with 14 arms height (8.33 m). It rises over the marble ring where the ribs of the dome converge with a diameter of 5 fi (3.27 m). It has a drum with a perimeter gallery covered by a nearly flat annular vault that extends itself towards inside in the dome of the lantern.

4° Over the vault that covers the lantern, in the extrados, there were to be built other small spires joined by small arches to the central one projected by Croce, with 49 arms height (29 m). This one would be composed by eight slender pillars joined between them by a lattice masonry on the sides of the octagonal prism. There would be a stair inside the spire joined to the perimeter masonry by iron bars. The crown of the spire would be a massive pyramid and a great marble statue.

Boscovich

In 1764, Boscovich, Croatian jesuit and professor of Mathematics at the University of Pavia at that moment, received a letter from the people in charge of the Duomo of Milan. They asked him for an expertise about the stability of the spire projected by Croce that would crown the Tiburio, and also its repercussion on the global structure of the building. He delivered it on 24th February 1765 (Nava 1845, 13, 53–64), and it contained fifty five articles; Boscovich recognizes that he mainly applied the same method as in the expertise about the dome of Saint Peter, the famous *Parere*.[3]

Basic hypothesis

The survey of the dome

First, Boscovich studied in a detailed way the model of the projected spire and the model of the whole building. After this, he visited the dome with Croce's plans checking up the location of the different elements that the models showed. From these plans he deduced the main measurements and those that could be uncertain because of their small size were checked with Croce. Boscovich emphasizes the exactness of his procedure, that although it can not be like the accuracy of an astronomical observation, it is much more that the problem required (54).

Density of materials

Once measured the different elements, Boscovich calculated their weight. For doing this, he needed to know the density of materials, information that Croce also gave him. For instance, a marble cubic arm weights 800 pounds (2897 kg/m^3) and brick masonry with lime mortar, 600 (2173 kg/m^3).[4]

Iron rings

Boscovich also noted down the size and location of the iron rings that encircled the dome and calculated their resistance according to the values obtained by "the good experts".[5] He took into account the principle applied in the expertise about the dome of Saint Peter to calculate the resistance of an iron circle bar, six times greater than the resistance of a right bar and mentions Poleni, who confirmed three mathematicians' finding by means of the experiment of the octagonal thread.[6]

Virtual work principle and the properties of material

Boscovich applied the same tools used by him, Le Seur and Jacquier to evaluate the stability of the dome of Saint Peter. But we can notice that Boscovich considers a greater influence of the quality of materials when he applies the Virtual Work Principle.

> my inquiries are based on the one hand on infallible and evident geometrical principles and on the other hand, on the physical properties of the used materials, that only can be known through experience and careful observation.[7]

Inasmuch as geometrical principles, Boscovich considers the displacements that the forces, that is, the weights and the thrusts exerted by the iron rings, would suffer "in those masonry buildings where the resistances are lower than the forces that thrust and press down".[8] That is, he applies the Principle of Virtual Work, "basis of all the Mechanics applied to machines"[9] that states, as it is well known, that if a system of forces is in equilibrium, the positive and negative work that these forces would make for a virtual displacement of the whole system would be equivalent. "A force exerts a thrust as great as the velocity of its initial movement in its own direction" and so, the works are calculated "multiplying the absolute forces by those distances that express these initial velocities".[10] In the case of masonry elements, the relation between the works made by the balanced forces and by the unbalanced forces must be equal or greater than 1 for a safe structure.

Now then, Boscovich points out that the hypothesis of the problem are slightly different from the assumptions made in the case of the dome of Saint Peter, since the Tiburio of Milan has no drum and instead of it, it has the counterpart of the lateral naves of the temple, that can not suffer displacements.

> So, I consider that this is a very essential difference between this masonry building and that one, difference that has me compelled to investigate a particular theory, that could be adapted immediately to this case, but also to other with due reflection.[11]

Although he does not mention it in an explicit way, there is an important difference: in this case there were no damages in the Tiburio to be explained, as in the case of Saint Peter's dome. In this case the scholars had to find out if the Tiburio would be damaged when the spire would be built. So, Boscovich had to apply his practical knowledge about the pathology of vaults and so, he established a hypothetical mechanism of collapse; in the case of the dome of Saint Peter the mechanism was deduced from the real damages. This process of abstraction led the three mathematicians to make some mistakes about the location of the hinges like the one situated on the keystone extrados that they located in the intrados.

In the second place, Boscovich speaks about the physical properties of materials. In the case of the Tiburio, he does not take into account the compression of the marble, the prevalent material. In return, he considered this compression in the case of the dome of Saint Peter at Rome, where the inner core of the drum

was brick masonry; compression that "increases a lot the force that thrusts and diminishes the resistance".[12] Boscovich recognizes that taking into account the compression of material gives very disfavourable results. Nevertheless, it does not seem that Boscovich considers wrong, twenty years later, to have included compression in the calculation of the stability of Saint Peter's dome, that are slightly wrong by this hypothesis, López (1998, 214–234; 1998b).

He also explains that it is easy to check in an experimental way the marble toughness and its resistance to compression. The test consisted in placing one or two small columns of marble standing on an horizontal plate of the same material, imposing on them an increasing weight, and later, measuring from time to time the height variations in the test specimens to check if they had compressed or not. Boscovich affirms that the columns would not compress even if lateral fissures appeared, and this justified his hypothesis.

In relation to the materials, but rather with collapse mechanisms of vaulted structures and his observations on real structures, Boscovich deals with the formation of hinges and sliding:

> When the masonry structures resent, this never happens without openings or detachments of one part with regard to the next one. That is, a surface slides along another one without leaving intermediate spaces or, that is more usual, the openings have a hinge form with an edge that does not slide along the contact plane. This can be deduced from experience, although it could be deduced also from theory, that is, from the nature of things, but I would be long-winded too much if I would start to explain all the principles that have guided me in my investigation and prove that are according to the known laws of the same nature.[13]

As Benvenuto (1991, 371–374) states, Boscovich got ahead of Coulomb (1773), who will expound in a methodical way the possible ways of collapse caused by sliding and formation of hinges (Heyman 1972). Monasterio, a Spanish engineer who analyzed Coulomb's work in a more detailed way in XIXth century, showed these mechanisms, figure 6.

Description of building and damages

Later on, Boscovich describes the building, specifically the crossing area over which the Tiburio raises. He distinguishes four basic levels: the first one until the springing of the dome, with the main arches and ribs that concentrate the efforts towards the main piers; over this part, the dome over an octagonal plan, that at the base springs from a false drum and it is composed by eight ribs or transverse walls that act as counterforts. Between these ribs were built the inner pointed

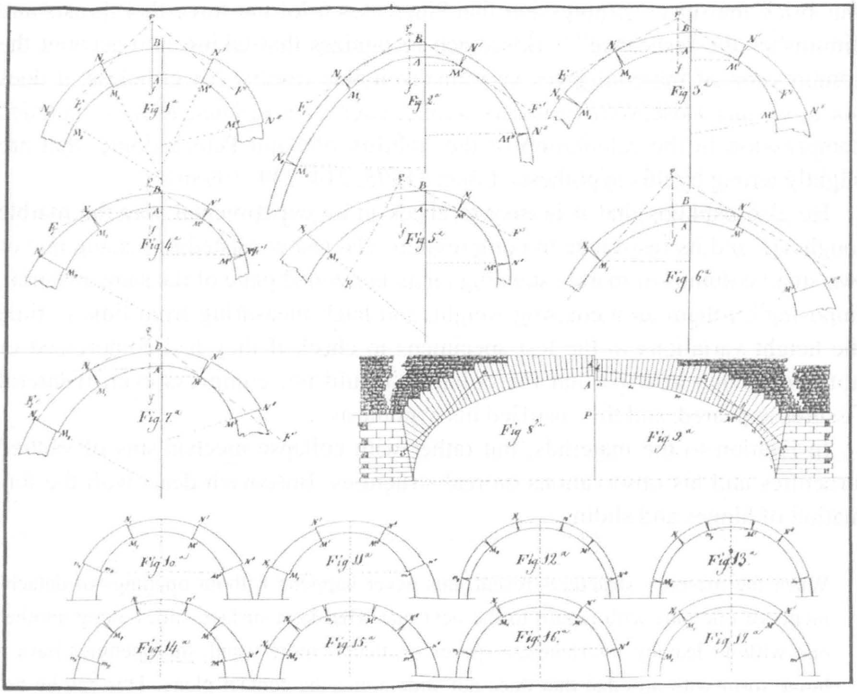

Figure 6
Different mechanisms of collapse by formation of hinges and sliding according to Monasterio (Huerta 2004)

shell and the nearly horizontal shell located outside; the third level corresponds to the lantern and the fourth one to the projected spire. He also describes the location and dimensions of the two octagonal rings that encircle the dome, and tells that the joints between the great marble pieces of the ribs converge on the centre of curvature of the dome. This detail is interesting to understand the collapse model that he will consider when calculating the stability of the dome (56–7).

Besides, although the aim of the expertise had no relation to the damages of the dome, Boscovich describes the light damages of the building just in case they would have influence on the stability. So, the masonry was undamaged except for one of the spires at the base of the dome because of a thunderbolt. At the base of the drum he could see some vertical fissures, with kite tail witnesses in the area next to the damaged spire, some of them broken and other sound. Boscovich also surveyed the dome from the inner side through the windows and he only discovered some detachments of the coating that let see the brick masonry. Finally, the people in charge of

the building told him that at the end of the building, near the cemetery, there were damages caused by a slight settlement of the foundation, but these did not have an effect on the vaults; this means that the works made until that moment had stopped the movement and, besides, the dome was far away from this area.

Calculation of weights and the thrust resisted by the rings

With the same criterion as in the building description, Boscovich calculates the weight of the four levels of the building, starting from the spire, and the thrust resisted by the iron rings.[14] It can be calculated that the repercussion of the weight of the spire over the total weight that support the foundation is 240.000 for 14.352.000 pounds, that is, 1/60, that can be neglected.

1° Spire projected by Croce	200.000 pounds	(65.360 kg)
The small spires at the base and the small arches that would join them	40.000 pounds	(13.072 kg)
Total	240.000 pounds	(78.432 kg)
2° Lantern	336.000 pounds	(109..805 kg)
Marble ring that acts as a base and join the ribs of the dome	40.000 pounds	(13.072 kg)
Total	376.000 pounds	(122.877 kg)
3° Dome itself, with drum, ribs and shells without the ring of the oculus	4.136.000 pounds	(1.351.645 kg)
(Without adding the filling and lattice of the windows and, by other side, without discounting the voids in the ribs)		
4° Piers	4 × 800.000 pounds	(1.045.760 kg)
Main arches and masonry from the impost till the springing of the dome	4 × 1.600.000 pounds	(2.091.520 kg)
Total aprox.	10.000.000 pounds	(3.268.000 kg)
5° Rings (each one of them)		
Resisted tension	164.000 pounds	(53.595 kg)
Total Horizontal thrust for an octagonal profile	1.131.000 pounds	(369.611 kg)

Table 1
Total weights and resisted thrust by the rings of the Tiburio of Milan

The stability of the spire itself

Once calculated the weights, Boscovich starts his analysis of the stability of the building studying the spire itself. Mainly he deals with the concept of resistance, although without giving concrete quantitative values.

First, if the base over which the eight small pillars of the spire rise, does not give up, these pillars are able to resist their own weight, just in the same way as the four great piers support the Tiburio without cracking. Inasmuch as their slenderness, it has to take into account the masonry and the iron bars that will join together and to the central stair, respectively, so that the proportion between the height and the width of the spire is actually between 5 and 6. According to

Boscovich, there are a lot of more slender towers like, for instance, the minarets that he could see in the mosques of the eastern countries.

After this, the base of the spire, that is the lantern, has to be considered. There is no lateral thrust because the spire pillars correspond with the inner pillars of the lantern and also the small spires that encircle the great one correspond with the outer pillars of the same lantern.[15] Only the inner part of the spire that rests upon the small lantern dome could suffer some damage. Now then, the false dome must open also towards outside to make possible this sliding and this is very difficult because of the weight that the lateral parts of the dome support.[16] That is, although it could be supposed that this dome act as a vault with lateral thrusts, because of the presence of a vertical key between the other horizontal courses, the counterparts are much greater proportionally and would prevent from overturning and subsequent sliding. In any case, Boscovich recommends putting granite slabs of great thickness that serve as a base for the spire to compress in a homogeneous way the lantern, without horizontal thrust.

Collapse mechanisms in vaulted structures

Before applying the Principle of Virtual Work to the dome it is necessary to choose in a correct way the possible collapse mechanism. Boscovich enumerates the different cases that can be found:

1° Arches and vaults. When they support an important weight on the key stone, they use to crack in the intrados at that zone, "remaining joined together the upper part like a hinge or a ball-and-socket joint. The same arch opens in the extrados, usually at a distance of one third, moving towards outside, in such way that the outer part cracks and the inner one remains joined as if there were a hinge there".[17] Finally, a third hinge appears at the base of the support, that turn around its extrados edge, but that, Boscovich explains, "does not use to move horizontally"[18] because of friction; for this to happen it would needed a horizontal force three times greater than the vertical one that acts upon the support.

> Therefore, the support, with a third part of the arch, turns towards outside around the immobile extrados angle at its base, and the two thirds of the upper arch turn with its bottom part towards outside and with the upper one that descends vertically, until finally the leaning supports overturn and falls the upper part of the arch divided in two parts with the whole weight that resting upon it was pushing it down.[19]

The placing of horizontal iron chains at a third height of the arch from the springing and also, the placing of counterforts prevent that the collapse happens.

2º Domes over drum without lantern.

> it happens that, for a homogeneous resistance, they open all around the perimeter like a shoot or pomegranate when the keystone of the vault descends and the rectilinear cylinder that supports it and it is usually called drum moves towards outside, which besides has to open vertically with cracks that widen when raising to the impost.[20]

3º Domes over drum with lantern.

As the ring over which the lantern rests is on the keystone of the dome, and resists compression, the cracks of the intrados appear slightly far away from here, in the transition between the dome and that ring and, so "the other opening has to produce lower down and near the impost".[21] Like the arches and barrel vaults, the dome yields out, and so a third hinge has to appear at the base of the drum, over which it will turn around and collapse.

For all the cases described, Boscovich explains that when the masonry is made of brick and lime the deformation usually consists of small compressions and dilatations of the material, instead of horizontal cracks, so that the structure bends instead of cracking. Indeed, the domes also show vertical cracks, although the horizontal cracks are easily unnoticed. This was the case of Saint Peter's dome.

4º Finally, Boscovich considers the particular case of the Tiburio, which dome with lantern has really no drum. He proposes two possible ways of collapse. The first one is based on the appearance of hinges in the keystone extrados, in the intrados of an intermediate area and in the extrados of the springing of the dome.

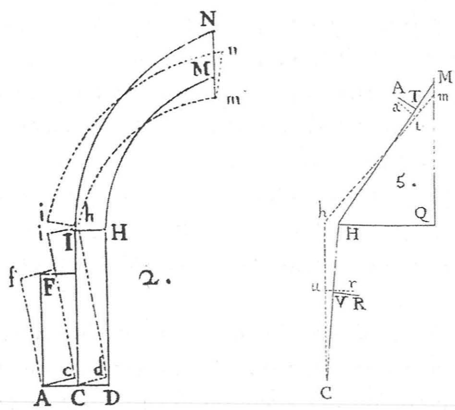

Figure 7
Hypothetical mechanism of collapse in Saint Peter's dome (Boscovich, Le Seur and Jacquier 1743)

The second one combines the presence of hinges with sliding: the ring of the keystone and part of the keystone detaches from the rest of the dome and this one, when turns outside around the extrados edge of the springing, lets the ring slide down with the lantern. It is the well known model of La Hire, but analysed in a different way and applied to a structure without buttresses or drum in this particular case (see figures 12 and 16 in figure 6).

> When the Dome has no drum over which its springing rests, but it raises at the same level as the lateral resistances that the naves exert, like happens in this case, can not occur the movement analogous to the mentioned above without an inner lower opening also in the arch, so that the bottom part of the same cracked arch turns around the extrados immobile edge, moving away its upper edge together with the lower edge of the upper part, and the upper edge of this one, opened towards inside and cracked makes descend with it the Lantern: or also one can imagine that instead of three hinges there are only two, one at the base of the impost, and the other one in the keystone next to the Lantern, remaining the extrados angle of the arch between these openings constant and turning outside its upper edge, while the Lantern, that has detached, descends through the space let by the opening of that edge as a receptacle of the wedge that, charged with the Lantern, descends with it.[22]

The application of Virtual Work Principle to the analysis of the dome

From the two mechanisms of collapse that we have just described for the domes with lantern but without drum, the first one he thinks it is not applicable to the case of the Tiburio.[23] The second one would be impossible if the joints between the marble pieces of the ribs that converge to the centre of curvature would have been more horizontal. But they are placed in a more vertical than horizontal position, and so it had sense to analyse this mechanism because it was possible that the lantern would break in the ring area and slide making the dome turn. Besides, he calculates the most disadvantageous position for this plane of sliding:

> I have found the way of calculating the forces that cause this type of movement and the resistances that stop it in the case that these same resistances are related to those forces with a minimum proportion and I have found the resistances much greater.[24]

The results obtained by Boscovich for the total weights are in table 2. As can be checked, the resistance forces produce a work four times greater than the unbalanced.[25] Boscovich draws our attention to certain factors that would increase even more that value: the toughness of the different parts, "which by itself has kept on foot the Dome of Saint Peter since so many years";[26] the work of the iron ring located over the windows at a distance of more than 30 arms height

	Unbalanced works	*Weights*	*Vertical displ.*
Spire and lantern	10.000.000 pounds	616.000 pounds	(16,24)
	Balanced works	*Weights*	*Vertical displ.*
Dome with the oculus ring	17.000.000 pounds	4.136.000 pounds	(4,11)
		Thrusts	*Horizontal displ.*
Lower iron ring	9.000.000 pounds	1.131.000 pounds	(7,96)
Upper iron ring	14.000.000 pounds	1.131.000 pounds	(12,38)
	40.000.000 pounds		

Table 2
Balanced and unbalanced works that result from the analysis of the Tiburio of Milan according to a collapse model of sliding

(17.85 m), that would have a value of 30 millions of pounds and, finally, the friction on the breaking plane between the lantern and the part of the dome near the keystone.

In any case, Boscovich, not in agreement with these calculations, checks that it would be a more disadvantageous mechanism if varying the location of the lower hinge and the sliding plane of the wedge. He also was anticipating Coulomb's memoir of 1773:

> Later I have considered what it would happen if these openings would appear in any other place instead of the keystone and the base of the ribs and I have found that, everywhere, the resistance would be greater in relation to the force than at the exposed places and considered in those calculations.[27]

The great stability of this dome allows Boscovich to understand why the gothic constructions have resisted in a better way along time than the old Greek and Roman buildings. In fact, the lantern was adding a double weight than the spire would add, without causing any problem in the dome. The slight damages were caused by thunderbolts or work imperfections, but the stability was too assured.[28]

Regi

P. Francesco de Regi was the other mathematician in charge of studying the stability of the spire. He was barnabite, regular cleric of St. Paul, and professor of Mathematics at the College of Saint Alexander in Milan, who signed out his ex-

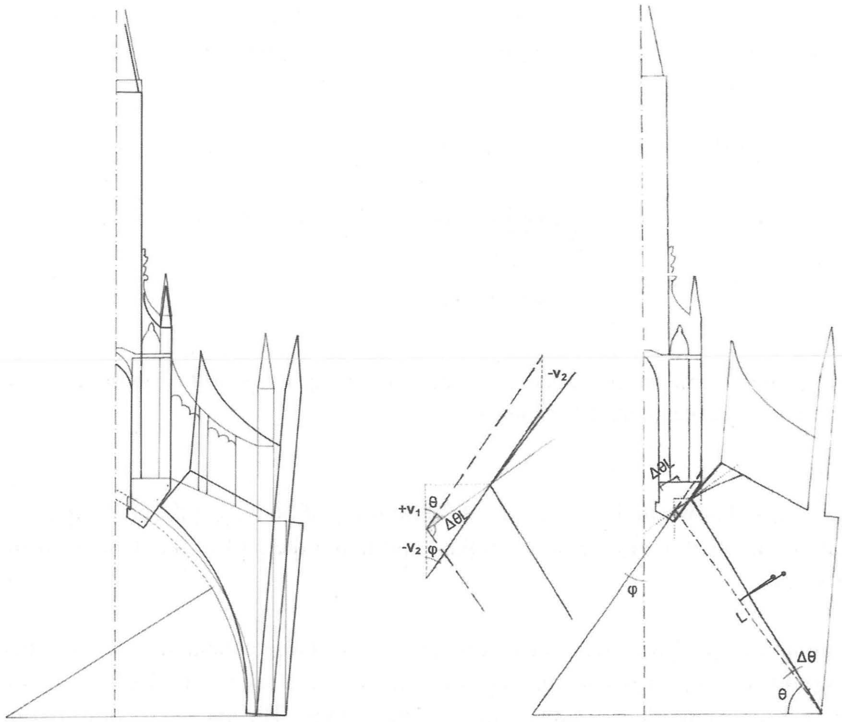

Figure 8
Hypothetical mechanism of collapse for the Tiburio of Milan analyzed by Boscovich. The dome profile is taken from Regi (1765)

pertise soon after Boscovich.[29] He analysed the stability of the spire projected by Croce and of the structure that had to support it: dome, main arches and piers.

The stability of the spire

First, Regi studied the stability of the spire, exactly the advantages of a pyramidal body compared to a prismatic one.

1° For the same slope and dimensions a prism overturns before than a pyramid, since that the gravity centre of the pyramid is lower, at a quarter part of the height from the base (see figure 1 in figure 9).

2° Regi studies the relation between the horizontal thrust that cause the overturning and the weight, both forces applied in the centre of gravity of the prism and the pyramid. The result obtained after establishing moment equilibrium is that the thrust that makes the pyramid overturn is greater, compared with the

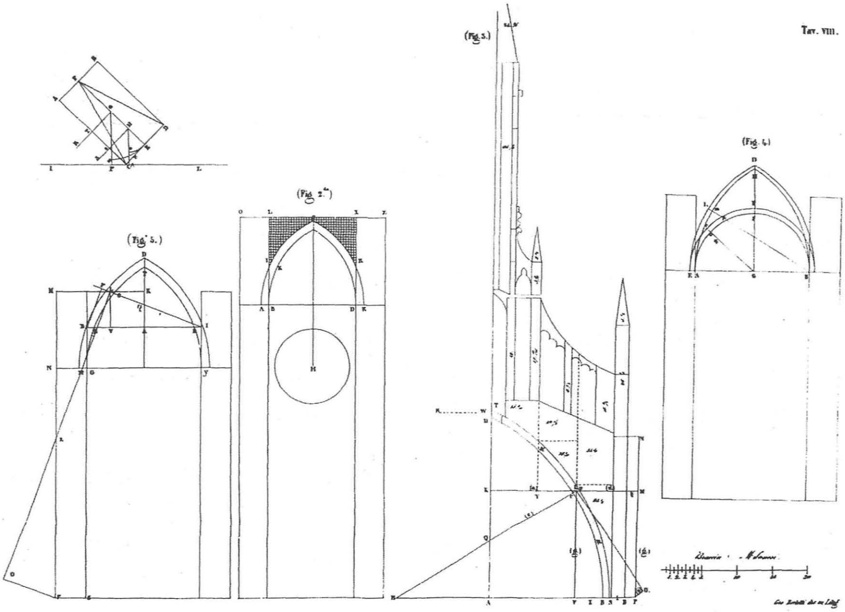

Figure 9
The stability of the Tiburio of Milan (Regi 1765)

weight, than in the case of the prism. That is, for a similar weight, the pyramid is more stable, since it requires a greater thrust to make it overturn.

3° Finally, Regi reflects on the real origin of the horizontal thrust that can suffer the prism and the pyramid: the wind. As the total force is proportional to the exposed surface and the incidence angle, the pyramid also is more stable because the wind thrust is smaller than in the case of the prism. (67) He states that the pyramidal volumes are especially suitable to counterpart the effect of hurricane winds that have an inverse figure than a pyramid. So, the projected pyramidal spire by Croce is very adequate, and it will be even more fit if the proportion between its base and height is greater.

The structure of the building: the main arches

After analysing the spire, Regi describes the Duomo. He insists on that the spire was foreseen in the original project and this was the reason why the building had the required structure to support its weight. Therefore a great number of iron ties were put through the naves and iron rings around the dome. In spite of it, he applied the theoretical principles to demonstrate that the different part of the

transept, main arches, dome and piers, could resist safely the weight of the lantern.

The first problem studied by Regi is "if the collapse of an arch can happen and therefore, the fall of a vault when a great weight is placed on it, supposed that the supports are firm and immovable, and that the arches are well reinforced on the extrados", figure 2 in figure 9.[30] That is, he analyses the resistance of the main arches as a separated part from the buttresses. His conclusions are the following:

1° The ashlars of the vault are resistant enough to avoid the whole arch can collapse because of lack of resistance (71–2).

2° It is impossible that the intermediate wedges of the arch can slide if the lateral stones remain immovable, since that their joints converge towards the centre of curvature and so, they expand in the extrados.

3° Nor the vault can deform itself in such way that the point C of the keystone approaches the point B at the springing, because the material "can yield only in a unconscious way".[31]

4° Finally, an opening at I that would separate IABK from ICK can not happen, because there is no space to let the masonry move, counterparted by strong supports.

Moreover, Regi gives as a proof of his statement a test made by him. He built a wooden model of a pointed arch (*terzo acuto*). He ordered that small wooden wedges were prepared that he after put in place between two wooden supports in vertical position over a table. He used flour mixed with cold water as a mortar and a small plank that has the function of centering. He also put the wooden filling over the extrados and the model stood up at the moment of decentering, in such way that he could put over a brick like a point force that did not alter the equilibrium. The resisted weight of the brick was 49 ounces (1.3 kg) for supports eight times more light, of 6 ounces (0.16 kg).[32]

To guarantee the obtained results, Regi mentioned the Duomo belfry, that in spite of the strong weight of its three bells, it was in a perfect state because of the strong counterpart of the main nave.

Analysis of the stability of the dome

The second problem raised by Regi is if the dome with its false drum is stable when considered isolated, which is, separated from the main arches and the piers over which it rises.

Although not in an explicit way, Regi calculated the stability of one of the eight ribs of the dome, that is, of the transverse wall that joins the inner shell of the dome with the drum. He considers that all the parts are made of a homogeneous material, although they are actually made of marble and brick, and that the

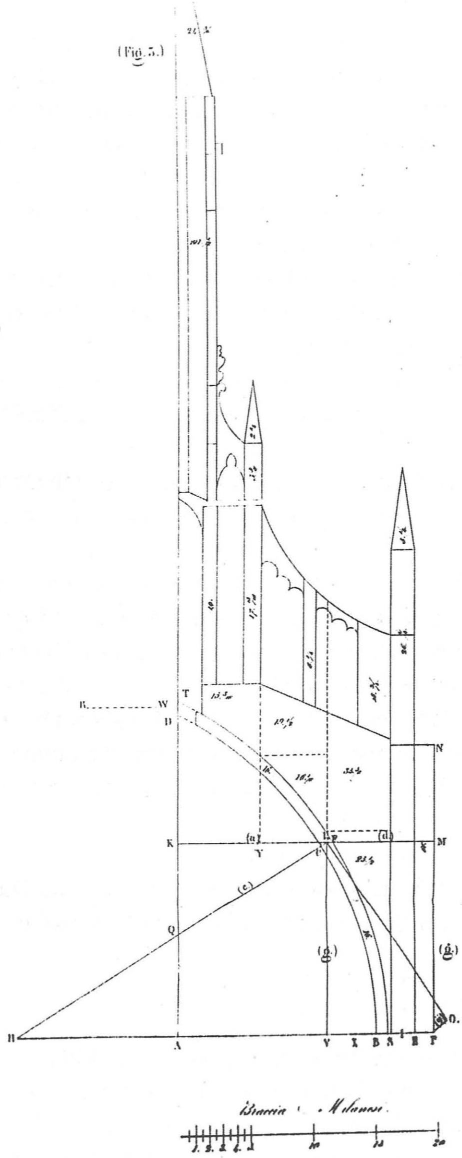

Figure 10
Analysis of the stability of the dome of the Tiburio of Milan (Regi 1765)

spire is massive. Instead of calculating the weights, he calculates volumes, taken as the surfaces of each element section, and he represented their value over the figure 10.[33]

The next step is to determine which parts of the dome thrust and which parts resist that thrust. He applied La Hire's method, according to Bélidor,[34] taking into account that the springing of the dome is at the same time the base of the ideal drum that overturns because of the thrust. So, from the centre of curvature of the intrados, H in the figure 10, it has to be drawed a line until the middle point of the arch BDST, that corresponds with the intrados nerve of the rib (also projection of the inner shell between the ribs).[35] The middle point of the joint of intersection, L, determines the vertical Lg that divides the section in two parts, on the left side, the part of the dome which weight causes the thrust and on the right side, the parts that counterpart it.

Once obtained the point L, it is drawed the perpendicular LO to the radius HL, that will be the direction of the thrust and after this, the perpendicular PO to LO from the extrados edge of the drum, that it is the lever arm of that thrust. The value of this thrust is obtained dividing the weight of the elements of the coin in two components, the horizontal one WR that acts on the keystone, in the middle of its width, and the one with direction LO, that is the one is being found out. Multiplying the thrust by the distance PO it is obtained the unbalanced moment. Inasmuch as the value of the balanced moments, it has to be calculated the sum of the products of the weight of each resistant element by the distance of P to the vertical that passes through its gravity centre.

Regi sets up certain geometrical relations to determine all those values and obtains the moments for the real dimensions of the Tiburio, which radius HB measures 28 arms (16.7 m), the angle LHV, 32° and the width BS, 1 arm (0.59 m.)

Balanced moments = 641 3/8

Unbalanced moments = 179 36/100 [36]

The proportion between them is 3,58 and, so, the dome with the spire will go on being very firm. As Boscovich, Regi also obtained balanced moments greater than unbalanced moments.[37]

Analysis of the stability of the piers

Finally, after analysing the behaviour of the dome, Regi considers necessary to analyse also the behaviour of the piers (74–77). He applies again Bélidor's version of La Hire's method, but introducing some changes. According to Regi, this method is very adequate to analysis arches or barrel vaults, but not pointed arches, like the main arches. He explains that he built another wooden model of a pointed arch (*terzo acuto*) and a semicircular one to check that the centre of gravity of the first one (semi arch) is nearer to the support than the centre of gravity of

the semicircular one, figure 4 in the figure 11. This is the reason why the breaking joint that separates the part of the arch that thrusts from the part that resists with the counterpart, forms in different places in both arches.[38]

So, for calculating the stability of the buttresses or piers that support a pointed arch, Regi divides the height of the pointed arch in three parts. The upper part will be the wedge that thrust and the two other thirds of the arch will help the support to counterpart it. As it can be seen in the figure 5, figure 11, to determine the direction of the thrust it has to be drawn the line ICF, where I is the point of intersection of the inner edge of the opposite pier with the horizontal one that passes at a distance of 1/3 from the springing of the arch, and C the middle point of the arch ET. By the middle point L of the intersection of the line ICF with the arch it has to be drawn a perpendicular line LO to ICF. The distance PO

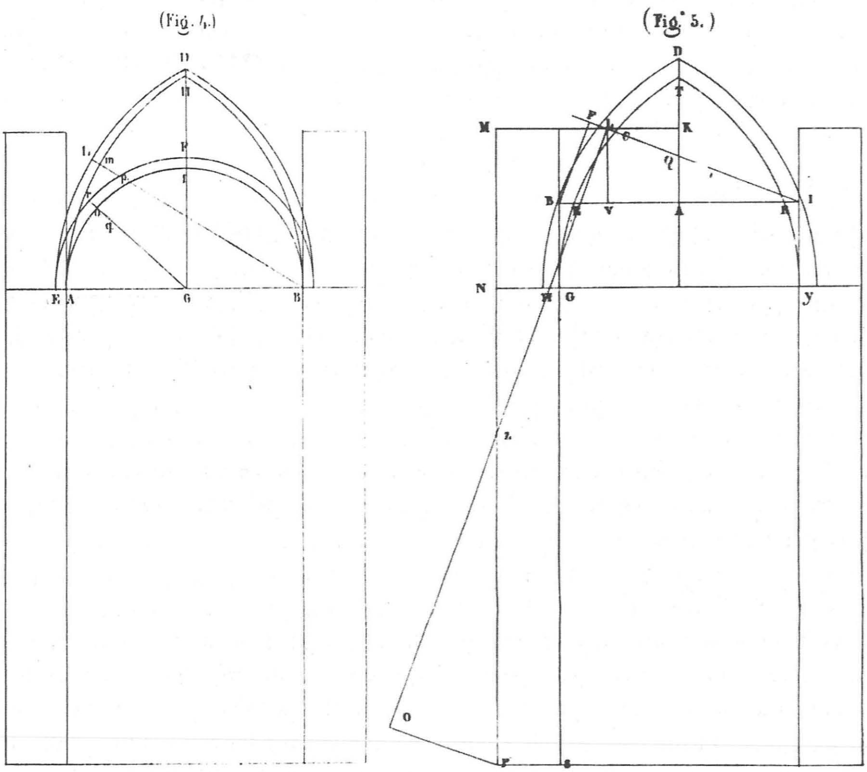

Figure 11
Analysis of the stability of the piers that support the Tiburio of Milan (Regi 1765)

from the extrados edge of the base to LO gives the lever arm of the thrust, which value is obtained like it was described above, decomposing the weight of the upper third of the semi arch in a horizontal component and other parallel one to LO. Once obtained the unbalanced moment, the balanced one is calculated adding up the moments produced by the weight of the pier and the 2/3 of the bottom part of the arch in relation to the edge P.

Regi does not give numerical results of the analysis of the piers because he admits that he did not have had time to take note of the needed information. Nevertheless, he gives approximate calculations about the stability of the piers loaded with the spire, the dome and the main arches, in the choir direction. Taking into account the weight of all the spires, the relation between the unbalanced and balanced moments is 6633/6884, that is, the structure is stable; but if the spires are not considered, the relation is worse, 6633/5988. In any case, the structure is not in a dangerous state, local nor global, since that friction and numerous iron ties of the naves are not considered.[39] He recommends that the spires and decorative walls are built as soon as possible and in this way they can help with their weight to increase the stability and also, that the arches of the naves are reinforced in the nearest area to the dome.

Conclusions

Boscovich, a famous scientist of his time, contributed to the development of the scientific theory of domes in a very remarkable way. He took part in the debates about Saint Peter's dome, the Bibliotheca Caesarea's oval dome and the Tiburio of Milan and anticipated some basic concepts to understand the behaviour of masonry vaults and domes, for instance, the possible mechanisms of collapse that Coulomb would explained in 1773 and the maximum and minimum method to find the real one. He was a theoretist but also a practitioner in the sense that he studied real cases and his theory was developed in this context. Although the famous *Parere* was written also by Le Seur and Jacquier, it seems clear that Boscovich played the most important role in the analysis of the dome. In contrast to Poleni, who applied for the first time the safe theorem or equilibrium theorem to the analysis of Saint Peter's dome, Boscovich studied domes from the point of view of a possible collapse, like La Hire, but making a careful study of the damages in real structures. So, he analyzed a real structure for the first time, Saint Peter's dome, applying the very powerful Virtual Work Principle to a simplified model of the dome. In the case of the Tiburio, he applied the theoretical principles to project a new structure, and proposed a new hypothetical mechanism of collapse. Nevertheless, he had little influence on his contemporaries and his method is not comprehensible to the scholars, even by Gauthey in his analysis of Saint Peter's

dome in 1998. On the other side, Regi also studied the Tiburio in a scientific way applying La Hire-Bélidor's method for the first time to a dome, but not in a correct way, because he only analyzed the ribs as arches. This method will be used in the second middle of the XVIIIth century to analyze the dome of Sainte Geneviève, in Paris, by Bossut, Patte and Gauthey and it will be the reference to check the validity of the scientific theory of arches and vaults until XIXth century (Huerta 2004; Huerta y Hernando 1998). In this context, Boscovich's contribution shines as an inspired one and so, he can be considered as one of the fathers of the scientific theory of structures.

Notes

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1. An arm in Milan is 0.5949 m. It contains 24 ounces and 144 points, like in Florence. Parsons ([1939]1976, 625–640).
2. We have considered here that a point is equivalent to one twelfth of an ounce, that it is, as in Parma, one twelfth of a Milan arm. So, according to Parsons ([1939]1976, 630), a point is equivalent to 4 mm. We considered before that an arm in Milan is equivalent to two palms, as in Florence, and so, 24 ounces, according to Regi (1765, 69), but in the case of the iron rings, whose size gives Boscovich, would have a too small section, 2,25 x 3,75 cm^2.
3. Boscovich was born in Yugoslavia in 1711 and died in Milan in 1787 (see references about his biography and works). Along the expertise there are notes relating to a hypothetical attached document containing six articles where some questions would be explained in detail: a plan and two cross sections with elevation marks, weights calculation, the resistance of the iron rings, cracking patterns for arches and vaults and the analysis based on Virtual Work Principle. Nevertheless, in Nava's transcription this attached part is not included and the Archive of the Milan cathedral has confirmed us that it is not there. Perhaps Boscovich did not want to publish this part.
4. According to Parsons ([1939]1976, 630 y 636), a "*braccio*" in Milan was equivalent to 0.5949 m and a pound of 12 ounces weighed 0.3268 kg but like Boscovich mentions great pounds of 28 ounces, we have deduced that their weight was equivalent to 0.7625 kg.
5. "da buoni sperimentatori" (54).
6. Benvenuto (1991, 356) states that Boscovich was probably the mathematician who established the relation between the resistance of a straight iron bar and the resistance of a circular one, first exposed in the *Parere* at the end of 1742.
7. "le mie ricerche in parte sono appoggiate a principj geometrici infallibili ed evidenti ed in parte alle fisiche proprietà delle materie adoperate, le quali non ponno conoscersi, che colla esperienza e diligenti osservazioni". (55)
8. "nelle fabbriche nelle quali le resistenze sono inferiori alle forze spingenti e prementi". (54)
9. "fondamento di tutta la meccanica applicata alle macchine". (54)

10. "una forza esercita un conato tanto maggiore, quanto sarebbe maggiore la velocità del suo moto iniziale secondo la sua direzione, se vincesse, o contro di essa se fosse vinta; onde si ricavano i movimenti moltiplicando le forze assolute per quelle lineette, che esprimono queste iniziali velocità". (54)
11. "onde questa la stimo una differenza essenzialissima tra questa fabbrica e quella, la quale differenza mi ha costretto a cercare una teoria particolare, che si potesse adattare immediatamente a questo caso, benchè pur si potesse colle meditazioni dovute trasportare ad altri casi". (55)
12. "quale compressione accresce molto la forza che spinge e diminuisce la resistenza". (55)
13. "quando le fabbriche patiscono, ciò non succede mai senza una qualche apertura o distacco di una parte rispetto alla contigua. Non succede mai senza che una superficie si strisci lungo l'altra senza aprirsi per uno spazio di mezzo, e per l'ordinario ciò succede senza che neppur una punta strisci su di un piano, ma l'apertura si fa solo a modo di cerniera. Questra cosa si ricava dalle esperienze, ma si potrebbe ancora dedurre dalla teoria, ossia dalla natura delle cose, ma io mi dilungherei troppo se mi mettessi ad esporre tutti i principj, che mi hanno guidato nelle mie ricerche ed a provarne la conformità alla leggi conosciute dalla natura medesima". (55)
14. Boscovich adds the weight of the oculus ring to the lantern (58). Later, he gives the exact value of this weight, 40.000 pounds (60). Also it can be deduced that Boscovich multiplied by 6,9 the tension force exerted by the iron rings, which section is equal to 11×18 points^2 (4.5×7.5 cm^2). This is excessive because in an octagon the proportion between the total force that pull out the angles and the tension in each side is equal to 6.08, and in any case smaller than the proportion corresponding to a circular one, 2π. Nevertheless, he considers a resistance of approximately 1588 kg/cm^2. It is possible that he considered a tension resistance greater than 164.000 pounds, and this would explain the oversized horizontal thrust. In the expertise over Saint Peter's dome the iron rings had a section of 3×4 ounces^2 (5.6×7.4 cm^2), similar to the Tiburio's rings, but there it was considered a resistance of about 1713 kg/cm^2.
15. The small arches that would join the main spire to the perimeter smaller spires have a negligible horizontal effect. (60).
16. I speak of a false dome because this was built with horizontal courses, as it was already described. The collapse mechanism by sliding of the keystone is similar to La Hire's model, but Boscovich speaks about a breaking joint that appears before the sliding movement, that is, there is friction between the elements but this friction can be surpassed and breaking areas can appear. When the supports turn out, the keystone can slide inwards. La Hire does not explain the origin of the breaking joints in his model (Huerta 2004; Huerta and Hernando 1998).
17. "venendo giù unita la parte superiore a modo di una cerniera. Si apre lo stesso arco esteriormente per l'ordinario verso il suo terzo, dando ivi in fuora, in modo che la parte esteriore si distacca e l'interiore va unita come se la cerniera fosse ivi". (60)
18. "la quale non suole dare in fuori orizzontalmente". (60)
19. "Quindi il pilastro con una terza parte dell'arco gira in fuori sull'angolo esterno immobile del fondo del pilastro, ed il pezzo di due terzi dell'arco superiore gira colla sua parte inferiore in fuori e colla superiore scende verticalmente rovesciandosi ai fine i pilastri caduti in fuori, e cadendo a piombo la parte superiore dell'arco divisa in due pezzi con tutto il peso, che aggravandola la spingeva in giù". (60–1)

20. "conviene inoltre in parità di fortezza, che si aprano da tutte le parti intorno a modo di cannocchia, o di mela granata nell'abbassarsi la cima della volta e dare in fuori il cilindro rettilineo che la sostiene e suole chiamarsi tamburro, il quale inoltre deve aprirsi verticalmente con aperture che, andando in su verso l'imposta, si slarghino". (61)
21. "l'altra apertura dovrà farsi più giù e più vicina all'imposta". (61)
22. "Quando la Cupola non ha tamburro in cima al quale essa sia impostata, ma nasce immediatamente al pari delle resistenze laterali, che oppongono le navate, come accade qui, non può darsi il movimento analogo al detto di sopra se non col fare che la più bassa apertura interiore segua ancor essa nell'arco, sicchè una parte inferiore dell'arco medesimo apertosi ivi giri intorno il cantone esterno immobile, andando infuori la sua cima insieme col fondo dell'altra parte superiore e la cima di questa apertasi di dentro e distaccato al Cupolino scenda giù con esso: oppure si può concepire, che invece di tre distacchi ve ne siano due solamente, uno in fondo verso l'imposta, e l'altra in cima verso il Cupolino, rimanendo l'angolo esterno dell'arco che sta tra le medesime aperture al suo luogo e girando in fuori la sua cima, mentre il Cupolino che l'ha cacciato in su, viene giù pel luogo lasciatogli dall'apertura di quel come ricettacolo del cuneo, che, aggravato da esso Cupolino, discende con esso lui". (61–2)
23. In a dome without drum it is possible a collapse mechanism by formation of hinges (Heyman 1995, 43), but in this case it is not possible because of the width of the ribs that support the lantern, that is, we can draw a rectilinear line of thrust inside the ribs.
24. "Ho trovato modo di calcolare le forze che agiscono ad indurre un tale movimento e le resistenze che lo impediscono nel caso in cui le resistenze medesime hanno il minimo rapporto a quelle forze ed ho trovato le resistenze assai superiori". (62) It is the first time that a scientists applies the minimisation method to find out the most disadvantageous mechanism of collapse, that is, the location of the sliding joint in the keystone area for which the resistance is the smallest one compared to the unbalanced works, taking into account that the joints converge in the center of curvature and the base hinge is immovable (Fig. 8).
25. Boscovich explains later that he included in the balanced works the weight of the eight spires that would be built over the drum counterforts. But this work would be 100.000 pounds and he underrated 200.000 pounds in the whole resistance of the dome to round off the work value to 17.000.000. Besides the iron rings work had to be added (63).
26. "la quale sola ha tenuta per tanti anni in piedi la Cupola di S. Pietro". (63)
27. "Ho poi considerato che cosa accaderebbe se queste due aperture invece di farsi in cima e in fondo a'costoloni si facesse in qualunque altro luogo, ed ho trovato che dappertutto la resistenza sarebbe maggiore rispetto alla forza che ne'due siti esposti e considerati in que' calcoli". (63) See also Benvenuto (1991, 374) about the significance of this trial and error method.
28. As in the *Parere* about Saint Peter's dome, Boscovich gives balanced and unbalanced works in pounds, but they should be multiplied by a linear measure. Boscovich does not give the displacements value in a explicit way (as he did in the *Parere*). I have deduced these displacements dividing the virtual works by the corresponding weights, and I have compared them with the displacements that would suffer an hypothetical mechanism like Boscovich's one and they are in the same proportion (Fig. 8). These are the relations between the displacements: $v_1 = \Delta\theta L \cos\theta$; $v_2 = \Delta\theta L \sin(\theta-\varphi)/\sin\varphi$ and $\Delta\theta L = K$. For $\varphi = 0.608$ rad and $\theta = 0.958$ rad, $v_1 = K\ 0.575$ and $v_2 = K\ 0.600$ (the

lantern with the spire movement); the gravity center of the ribs moves K 0.158 upward (I have simplified this part and considered the gravity center of the ribs, but not the gravity center of the rib with the two shells. It is not clear how Boscovich obtained this point because he calculated total weights and works as in the *Parere*. He does not explain if he considered 1/8 of the dome to calculate gravity centers). Rings movement is proportional to the height they are placed, Äèh, that is, Kh/L. (Fig. 8 is based on Regi's dome section, Fig. 10).

29. Nava (1845, 65–77) with 1 plate. According to Nava (1845, 13), this writing was delivered soon after Boscovich's one, on 10th March 1765.
30. "se possa succedere la rottura di un arco, e quindi la caduta de una volta per un grave peso sovrappostogli, supposto che gli sostegni sieno immobili ed invincibili, e che gli archi siano dalla parte convessa ben rinfiancati". (71)
31. "non può che insensibilmente cedere". (72)
32. It is possible that the paste used by Regi would help to resist the weight of the brick. He does not indicate the dimensions of the supports. (72)
33. Regi does not explain how the double shell built between the ribs is supported. From the numerical information I have deduced that he does not take it into account. That is, he analyzed the rib arches, that support their own weight and the lantern with one of the eight pillars of the spire. So, he considers the surfaces of the different elements instead of volumes.
34. Belidor (1729). Regi considers Belidor as "theorist and, also practitioner". (72)
35. Although all the rib resists the thrust, Regi places the thrust in relation to the granite nerve or projection of the inner shell.
36. They are not really moments because Regi considered surfaces and not weights, but they are proportional.
37. Boscovich analyzed the relation between balanced and unbalanced works, not moments.
38. Regi applied Bélidor's method to a pointed dome.
39. The lack of accuracy is about the calculation of gravity centers and the division between resistant and no resistant elements. Regi also explains that he did not take into account the heterogeneous resistant, that is, the voids that would diminish the balanced moments. (76–7)

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Towards equilibrium in Finite Element Analysis

Edward A. W. Maunder

Developments of finite element methods (FEM) since the pioneering days of the 1950s have focussed almost exclusively on displacement models, using concepts associated with element nodes and corresponding displacement fields. The earliest work usually recognised in the engineering community is by Turner et al (1956), where triangular elements for planar problems are proposed using a direct engineering argument for the formulation of the problem. A good account of the early developments can be found in Robinson (1985b) in résumés of the lives of the pioneers in FEM.

Displacement models have been dominated by conforming elements based on assumed displacement fields and a hierarchy of geometric entities which start with nodes and progress up to 3D domains representing different forms of structure —the commonality is the use of nodal connections with shared displacements and the concept of associated nodal forces as indicated in figure 1 for solid elements in 2D and 3D. The latter quantities are mathematical in nature rather than physical, although engineers are used to such in the form of resultants of distributions of stress or loads, e. g. self weight.

Such models satisfy two of the three conditions for an elastic solution, i. e. compatibility and the constitutive laws, but relax on the third condition of equilibrium. They are excellent for simulating deflected forms of structures, but we are left with weak forms of stress equilibrium represented by equilibrating nodal forces. In fact the more we derive quantities such as stress from displacements, the worse they become in terms of accuracy or equilibrium. A classic example of a failure due to this type of inaccuracy has to be the Sleip-

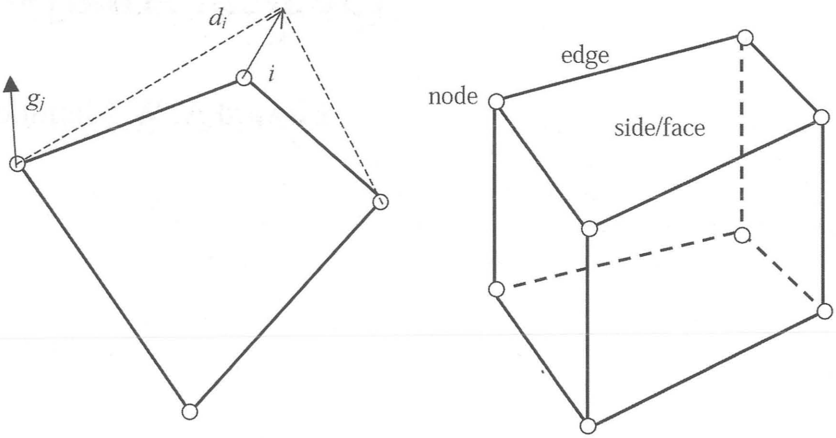


Figure 1
4 node quadrilateral and 8 node hexahedron

ner A oil platform in 1991 which is discussed in more detail later in this paper.

Parallel developments took place for equilibrium models in the 1960s, principally due to Fraeijs de Veubeke (1965), since it was recognised that solutions from both types of model complemented each other, and could provide bounds to solution errors in terms of energy norms. From the structural engineers' point of view equilibrium models would also provide the sort of stress information necessary to exploit the lower bound theorem of plasticity — otherwise known as the master safe theorem, Heyman (1995). However these developments appear to have been in the "slow lane", due partly to inherent complexities and partly to commercial resistance.

This paper takes a personal view of developments from displacement to equilibrium plate elements from the viewpoint of some 30 years of study, teaching, and use of FEM for structural analysis, and research into non-conventional methods with the emphasis on generating statically admissible solutions for stresses or stress resultants. It includes a number of illustrative examples and concludes with some considerations on the visualisation of stress fields beginning with the early work on photoelasticity by Maxwell (1850).

Displacement models

Alternative formulations

The formulation of finite element equations involving element stiffness matrices and equivalent nodal forces as loads has followed various arguments from those with direct engineering appeal to those based on mathematical functional analysis. Earlier texts by Przemieniecki ([1968] 1985) and Livesley (1975) extended the stiffness methods developed for skeletal structures with arguments based on virtual work and the unit displacement theorem which emphasized the concept of a nodal force. Zienkiewicz (1967, 1971) to Zienkiewicz & Taylor (2000) produced more general formulations including variational methods and methods of weighted residuals, and Strang & Fix (1973) presented mathematical analyses of these methods. The challenge has been to bridge the divide between engineers and mathematicians with sufficient mathematical rigour and yet maintain clear meaning in the engineering context. The latter methods did not need specific reference to nodal forces, however it transpired that these forces formed an essential key to a simple method which could recover stronger forms of equilibrium.

An alternative view of the weighted residual method was proposed by Maunder (1989b; 1991) which extended a direct formulation based on physical quantities to a general class of widely used conforming finite element models. This formulation recognised three roles for element shape functions: (i) to interpolate displacements from nodal values, (ii) to map elements to general curved shapes, and (iii) to disperse applied loads to statically equivalent nodal forces. The third role emphasized the nature of the weak equilibrium relations between nodal forces and internal stresses and external loads.

However the formulation is presented, stresses are derived from displacements with discontinuous derivatives and they do not satisfy equilibrium in a strong sense. Stress fields may be smoothed by an averaging procedure in a post-processing stage, and this may produce superficially more attractive results for stress, but on the other hand it may serve to hide their errors. The main problems for the design engineer as a user of commercial software has been that of facing uncertainty as to how the stress output has been derived, and of dealing with a lack of strong equilibrium when the safety of a design inherently depends on the master safe theorem. A worse situation exists when the designer is unaware of the lack of equilibrium, as appears to have been a major contributing factor to the collapse of the Sleiper A oil platform in 1991 (Jakobsen 1993; Kvamsdal et al 1993; Rombach 2004). An 8-noded hexahedral type of element was used which was both nonconforming and mapped into distorted shapes in a very coarse mesh. Consequently stresses were derived with poor accuracy and significant defaults in equilibrium.

Recovery of equilibrium

Equilibrium of nodal forces has been used to derive fully equilibrated stress solutions by Ladeveze & Leguillon (1983) and Stein & Ahmad (1977). The method of recovery is an inverse procedure to that of integrating stress fields to obtain nodal forces. It has three stages: (i) resolution of nodal forces into components associated with adjacent sides of elements, (ii) determination of statically equivalent side tractions which are co-diffusive, i. e. the components of normal and shear stress along element interfaces are continuous, and (iii) determination of stress fields within elements which are statically admissible with the side tractions. Stage (i) has been explained in simpler terms using Maxwell force and string diagrams (Ladeveze & Maunder 1996). This is illustrated in figure 2 where four elements, numbered 1 to 4, connect at node n and share interfaces denoted by i to l. A node force such as g_n^2 acts on element 2 and is resolved into components $(g_n^2)^j$ and $(g_n^2)^k$ associated with interfaces j and k. P is a pole point of the Maxwell diagram whose position is arbitrary. Sectional forces, such as transverse shear forces in the case of the Sleipner walls, can thereby be recovered as part of an equilibrating force system. Similar methods have also been proposed by Kvamsdal et al (1993).

In stage (iii) Ladeveze and Maunder (1996) advocated the use of displacement elements of higher degree higher to determine stress fields which are, for

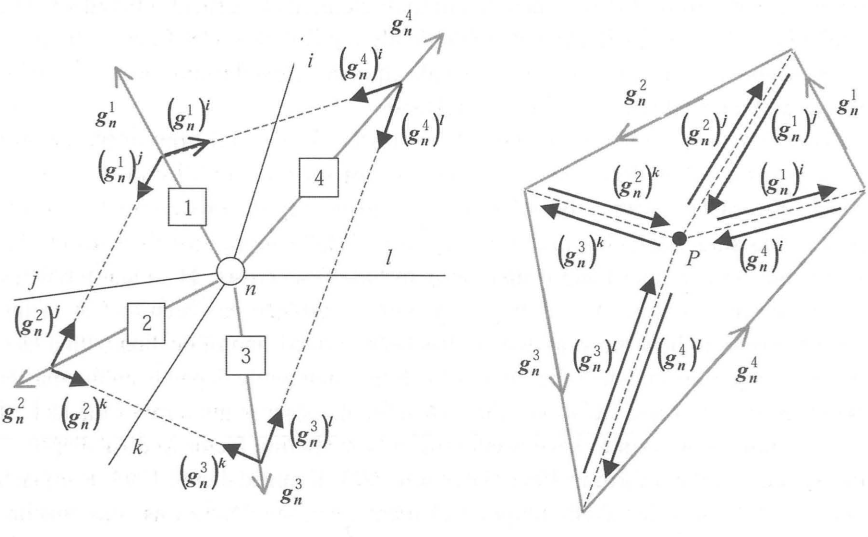


Figure 2
String polygon and Maxwell force diagram for a typical node

practical purposes, approximately statically admissible. However it is also possible to use stress based equilibrium elements at this stage in order to guarantee statical admissibility for a practical range of loading.

Equilibrium models

Developments for plane stress and solids

The development of equilibrium models begins with the seminal work of the Liege school beginning with Fraeijs de Veubeke (1965) and leading to numerous published works, many of which are reproduced in the Memorial Volume edited by Geradin (1980).

Stress based elements for plane equilibrium models were initially developed as a means of implementing the principle of complementary energy, and made use of Airy stress functions to define stress fields within elements. Connections between elements were enforced in two ways: (i) by identifying values of the stress function, and derivatives, at nodes and interpolating functions within elements; or (ii) via conditions on conjugate, or dual, modes of traction and displacement, or generalised forces and displacements, defined on the sides of elements.

Other proponents of the first way (Gallagher 1971; Sarigul & Gallagher 1989) formulated flexibility methods in terms of stress function variables. Such methods don't appear to have become popular due mainly to inherent difficulties with the application of force or displacement boundary conditions.

The second way was formulated into a conventional stiffness method, but with variables associated with sides of an element instead of its corner nodes. An inherent problem with such elements is the presence of spurious kinematic modes caused by an excess in the number of displacement modes over the number of independent stress fields. Such kinematic modes imply that certain modes of displacement, other than the rigid body modes, can exist as mechanisms and certain modes of traction cannot be transmitted through the element. The mathematical consequence is a rank deficient stiffness matrix. It appears that the only way to avoid this problem and maintain complete equilibrium was, and still is, to combine elements into macro-elements, and this was demonstrated by Watwood & Hartz (1968); Sander (1971); and Allman (1979).

A major report dealing with duality in linear and non-linear applications was produced for the US Air Force by Fraeijs de Veubeke, Sander, and Beckers (1972). The last paper by Fraeijs de Veubeke & Millard (1976) was produced in the year of the first author's untimely death, and there he explored a number of other possibilities for the future development of stress based elements having

various forms of equilibrium defaults. These included relaxing the continuity of self balanced modes of side traction, and the pointwise rotational equilibrium of a stress field, i. e. allowing non-symmetric stress tensors. The latter idea has been recently developed further by Bertoti (2002).

Robinson (1975; 1985a) reviewed the concepts of equilibrium models and gave more direct physical meaning to the modes of side traction. Maunder (1983) redefined these modes in a hierarchical way based on Legendre polynomials and classified them as basic, i. e. having a resultant, and higher-order, i. e. being self-balanced with no resultant. This classification, together with concepts from graph theory, was exploited in another form of flexibility method of analysis for models composed of stable macro-elements.

Although generalised displacements had been defined for the sides of equilibrium elements, hybrid concepts for stress based elements are generally attributed to Pian (1964). These elements used frame functions to interpolate boundary displacements from nodal values, together with simple internal stress fields which appear to have been selected fortuitously to be statically admissible (Pian 2000). Due to the nature of the displacement fields such elements do not enforce co-diffusive tractions and so they cannot form an equilibrium model!

A new wave of developments of hybrid/mixed elements was initiated at the Instituto Superior Tecnico, Lisbon, in the 1990s (Moitinho de Almeida & Teixeira de Freitas 1991; 1992). They defined hybrid elements with separate independent modes of displacement for each side and statically admissible stress fields. Their

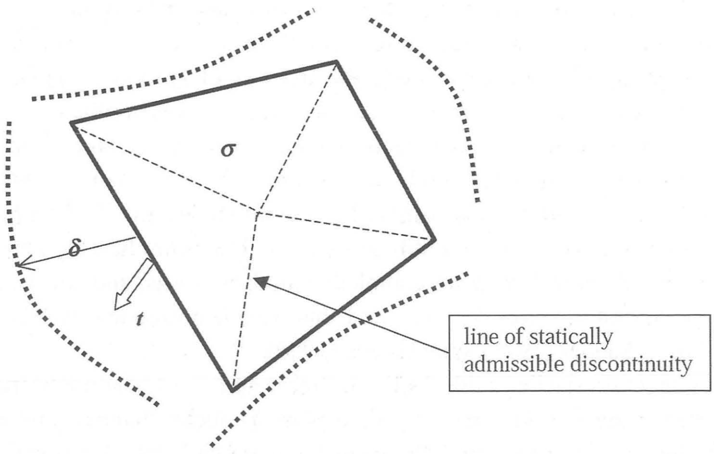


Figure 3
A typical hybrid element with side connection variables

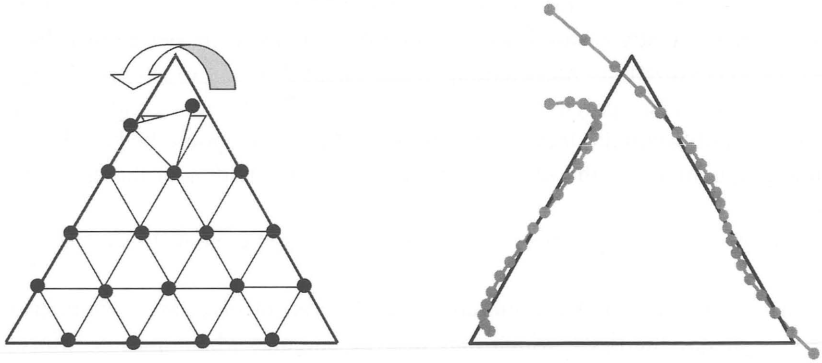
formulation allowed for the use of more general functions, as well as polynomials of high degree. Stress and displacement parameters $\boldsymbol{s}$, $\boldsymbol{v}$ are determined from simultaneous equations representing weak integral forms of compatibility and equilibrium. When polynomial functions are used, sufficient conditions for enforcing equilibrium are that the displacement functions should be based on complete polynomials of the same degree as the stress fields and the applied tractions.

Various methods have been employed for solving these equations, e. g.

(a) solve for $\boldsymbol{s}$ and $\boldsymbol{v}$ simultaneously, this can involve large symmetric but sparse matrices, which may also be singular when spurious kinematic modes are present in the model. Special techniques have been evolved to cope with this situation when the loads are admissible, i. e. the equations are consistent (Moitinho de Almeida & Teixeira de Freitas 1991);
(b) eliminate $\boldsymbol{s}$ and solve for $\boldsymbol{v}$ by a stiffness method of analysis (Sander 1971), again problems arise when spurious kinematic modes lead to rank deficient matrices;
(c) eliminate $\boldsymbol{v}$ and solve for $\boldsymbol{s}$ either by a flexibility method of analysis (Maunder 1983), or an integrated force method (IFM) proposed by Patnaik (1973; 1986).

As with the earlier forms of equilibrium models, the problems caused by spurious kinematic modes can be avoided by using the macro-element concept. The definition and visualisation of these modes depends on the formulation used. In the original formulation (Sander 1971) element kinematics were visualised by further subdividing a triangle into a system of triangular "skeletons" pin-jointed together with virtual nodes placed between the corners, e. g. as indicated in figure 4 (a) when degree $p = 3$. A spurious kinematic mode at a corner then becomes evident when the top part of the skeleton rotates as shown. On the other hand the more recent work on hybrid elements leads to the corresponding view of the corner mode shown in figure 4 (b) (Maunder & Moitinho de Almeida 2005).

When hybrid triangles are assembled into polygonal macro-elements, the macros become stable dependent on some simple rules concerning the polynomial degree p and the internal geometry. Quadrilateral macro-elements have been studied (Maunder 1989a; Maunder, Moitinho de Almeida and Ramsay 1996; Maunder & Moitinho de Almeida 1997) and their rules can be summarised as follows:



(a) $p = 3$ (b) $p = 3$

Figure 4
Spurious kinematic modes for a plane triangular element

(a) $p = 0 \Rightarrow$ constant stress fields: the macro is externally stable only when diagonal subdivision is used, but a single internal mode exists —it can be said to be benign since it can't be transmitted across the boundary of the macro;
(b) $p = 1 \Rightarrow$ linear stress field: stability rule as for (a);
(c) $p \geq 2 \Rightarrow$ higher stress fields: the macro is completely stable, both internally and externally, when subdivision other than diagonal is used. However when diagonal subdivision is used a single benign internal mode exists, as illustrated in figure 5 for $p = 3$.

These rules can be derived from either visualisation of the spurious kinematic mode in figure 4, although it appears that the greater level of stability for the higher degree elements in (c) was not appreciated by the Liege school (Sander 1971). Similar rules are presently under consideration for macros of general polygonal shape in 2D. Some studies have been made on polyhedral elements for modelling solid structures (Moitinho de Almeida & Almeida Pereira 1996), and it appears that a tetrahedral macro-element formed from four tetrahedra is free of spurious kinematic modes.

A numerical example takes us back to the investigations carried out after the Sleipner collapse, and illustrates the essential benefits of using equilibrium models, albeit with a very coarse finite element mesh. Figure 6 shows a plan view of the tricell region where the reinforced concrete wall section failed in shear at the position indicated, and the finite element discretisation used in the original analy-

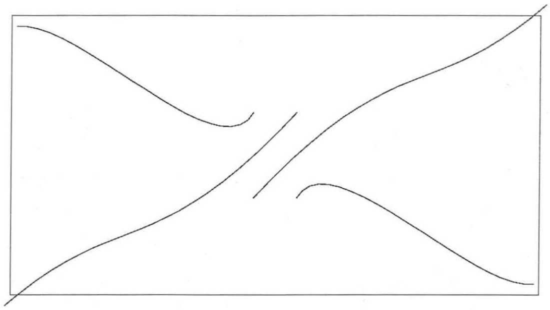


Figure 5
The internal spurious kinematic mode for a membrane when $p = 3$

sis (Jakobsen 1993). Solid elements were used in the original model, but for present purposes these are replaced by 2D hybrid equilibrium membrane elements of varying degree. The model has two axes of symmetry as shown where normal displacements are zero and tangential displacements are unconstrained. Loads are applied by a uniform water pressure of 670 kPa within the tricell and a com-

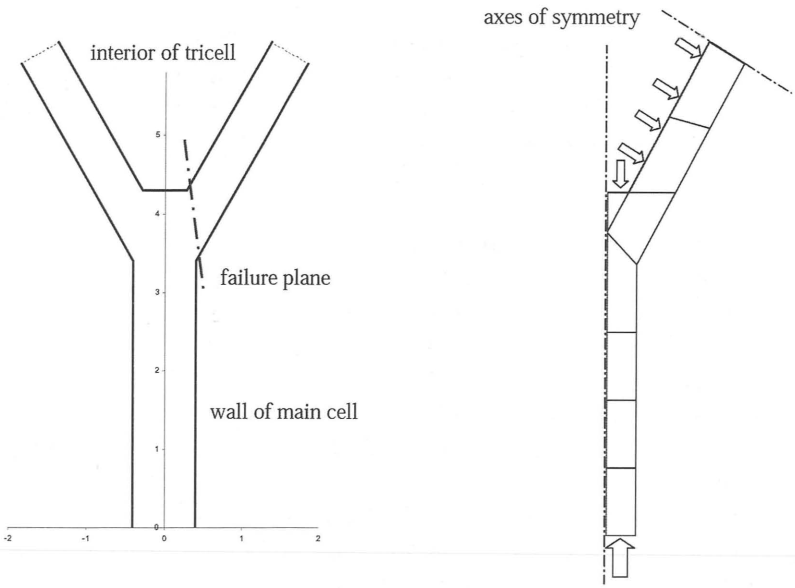


Figure 6
Sleipner tricell region and a symmetric finite element model

pressive force of 2,272 kN/m as a static boundary condition transferred from the main cell wall.

Of course, as has been noted by a Task Group of the Institution of Structural Engineers (2002), a structural analysis of such a substructure could be carried out using a simple 2D frame model. This would be perfectly adequate for evaluating stress-resultants, but wouldn't produce detailed information about the distribution of stresses in the crucial region where the walls join. Due to the symmetry involved, such a frame has only one degree of statical indeterminacy, the value of which would not effect the shear force on the plane where failure is believed to have occurred. This force amounts to 2,124 kN/m which corresponds to an average shear stress of 2.3 N/mm^2 on an inclined section devoid of shear reinforcement or compressive normal force! So failure in the context of reinforced concrete is understandable without looking deeper into the distribution of stresses. Linear elastic analyses of the equilibrium finite element model yields results for deflections of the model and tractions in the section where failure occurred in figures 7 and 8 respectively.

Particularly useful features of the results relevant to elastic behaviour include the following:

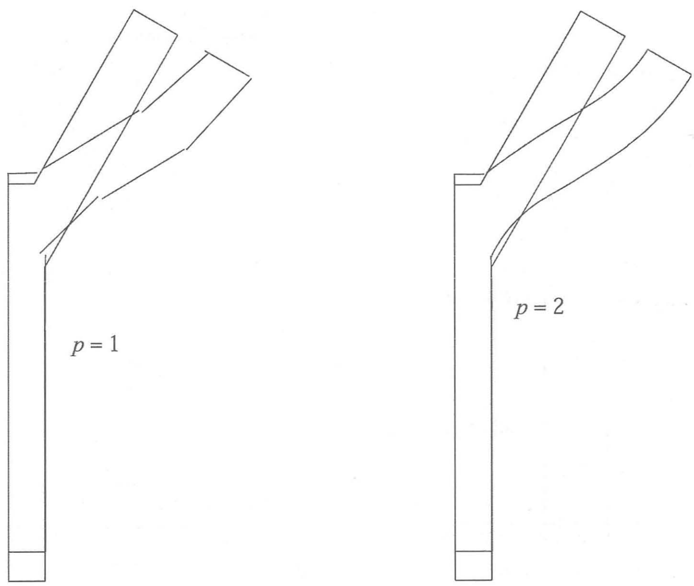


Figure 7
Deflections of tricell with degree p = 1 and p = 2

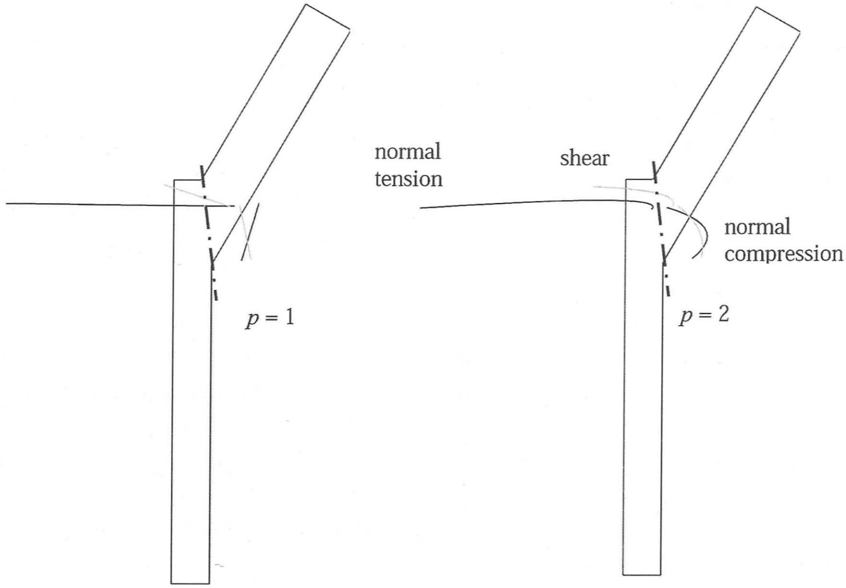


Figure 8
Distributions of normal and shear tractions on the failure plane

– a coarse mesh of elements having low degree can be sufficient to capture essential stress and deflection quantities for design;
– a simple graphical error indicator is provided by the lack of continuity of edge deflections;
– convergence in practice can be achieved by increasing the degree without changing the mesh, in the present example there are negligible discontinuities for $p > 2$;
– continuity of stress fields is improved by increasing the degree, but continuity is always sufficient for point by point equilibrium;
– points of stress singularity, such as at re-entrant corners, are of mathematical rather than physical significance. They can pollute the quality of a solution in their neighbourhoods, and careful modelling strategies may be required particularly for displacement elements. However the tendency with equilibrium elements is to "soften" such infinite stress concentrations in a similar manner to stress redistribution in the presence of plastic behaviour — which is exactly what engineers rely on in many practical cases!

developing the characteristics of hybrid triangular elements of general degree p (Maunder & Moitinho de Almeida 2005) which could be used in polygonal macro-elements to avoid the general problem of spurious kinematic modes.

Hybrid elements based on internal stress fields satisfying all three conditions for local elastic solutions, i. e. Trefftz fields, were developed by Jirousek & Leon (1977) to Jirousek et al (1995). These proved to be highly effective, but were not strictly designed to provide equilibrium models. It was shown by Teixeira de Freitas (1998) that these may be viewed as hybrid equilibrium elements when the boundary displacement functions are defined separately for each side. This case appears to be similar to that involving boundary traction modes proposed by Jirousek & Zielinski (1993). However the possibilities for spurious kinematic modes may be even greater when the internal stress fields are restricted to satisfy both equilibrium and compatibility of elastic strains!

Mixed and hybrid-mixed elements have also been proposed as routes to enforcing strong forms of equilibrium in the context of plates. Bertoti (2002). Pereira & Freitas (1996) defined a very general hybrid-mixed formulation for plate bending in which both stress and displacements fields are used internally, and independent displacement fields are defined on element boundaries. Strong equilibrium emerges provided equal polynomial degrees are used for these quantities, but again there would appear to be possibilities for spurious kinematic modes. Bertoti (2002) developed ideas originally proposed (Fraeijs de Veubeke & Millard 1976), and uses mixed fields of non-symmetric stress tensors and rotations in a version of the principle of complementary energy. The rotation fields are used to enforce symmetry of the stress tensors and thereby achieve complete equilibrium. The question of the existence of spurious kinematic modes is an interesting one, since the modes identified to date with hybrid elements usually correspond to rotations of corners of triangular elements!

An application of particular interest concerns the design and assessment of reinforced concrete flat slabs forming floor or roof structures, e. g. in multi-storey buildings and car parks. When the slab and its column supports are arranged in a regular geometric way, and only small openings are present, a common form of analysis represents the structure by an equivalent frame and empirical rules are used to design or check details such as punching shear forces in column zones. However when the form of the structure lies outside of common experience, e. g. when columns are arranged in an irregular pattern and large openings are present, finite element modelling comes to the fore. Of primary concern are the ultimate limit states (ULS) and appeal to the upper and lower bound theorems of plasticity via yield line and Hillerborg's strip methods respectively. The latter has the advantage of producing "guaranteed" safe designs or assessments for load capacity, but until now has been the more difficult to implement in a general

computational methodology as noted by Hillerborg (1996). The macro-element equilibrium model forms a convenient vehicle with which to explore statically admissible force paths through a slab and into the supporting columns via interfaces in the column zones. The generalised force paths involve the transmission of moments and shear forces defined by modes of traction on the edges of elements, and graph theory can again be exploited (Maunder & Savage 1994). Limit analysis based on these concepts has yet to be implemented, but linear elastic analysis has been implemented.

An example of the analysis of a column zone is now presented based on a flat slab for a commercial building designed for southern Portugal where the typical span is 8 m, and the column zone is formed by a 3 m square drop panel with a thickness of 0.4 m. The central column is square with side dimension 0.8 m. Poisson's ratio is taken to be 0.2. Two finite element models are considered based on hybrid equilibrium and conforming elements. The equilibrium model consists of 4 trapezoidal quadrilateral elements of degree 2 around the column perimeter, with a typical one shown shaded in figure 10. The conforming model, shown by the finer mesh, is based on 8 noded Reissner-Mindlin plate elements. Various ways may be used to model the column support (Rombach 2004), in this example the column is assumed to have a rigid cross-section and to be very stiff so that no rotations of the cross-section take place. Consequently strong singularities in the theoretical solutions for moment and shear exist at the corners (Rössle & Sändig 2001, Huang 2003). Along the sides of the column the displacement modes of

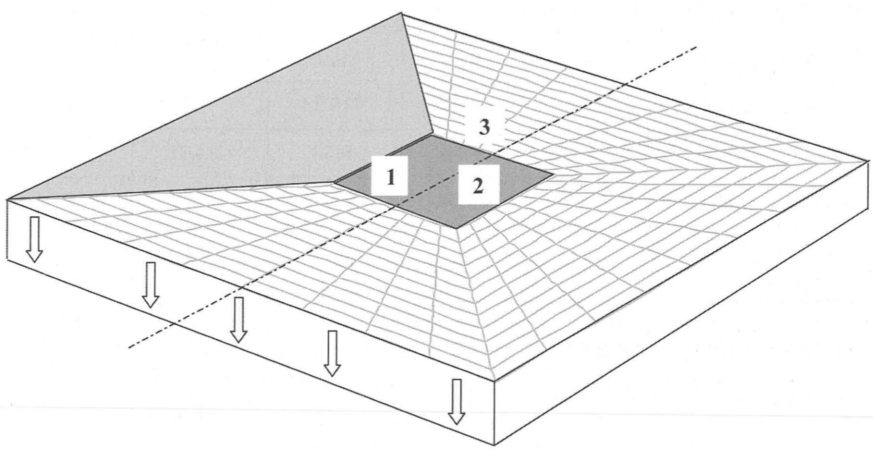


Figure 10
Finite element models of a column zone

the equilibrium model are restrained with zero values, and the nodes of the conforming model are fixed to represent hard supports.

The models were analysed in order to study the effects of "unbalanced" loads when a moment is transferred into the column, e. g. typical of uneven spans or loads, or when horizontal forces occur due to wind loads or seismic loads in an unbraced building structure. Loads are applied along one external edge of the drop panel as shown in figure 10. The first load case (i) involves a uniformly distributed shear force of 120 kN, and the second load case (ii) involves a linear distribution of shear forces equivalent to a torsional moment of 180 kNm.

The results in table 1 for total strain energy and stress-resultants on the sides of the column show good agreement generally. Figures 11 and 12 indicate some differences in the distributions of shear forces, however comparisons are complicated by the need to interpret results from the conforming model. For this model the linear modes of equilibrating shear traction have been derived from the nodal reactions in a similar way to that described by Ladeveze & Maunder (1996), with the assumption that the forces on the corner nodes of the column are split in half and the separate components are assigned to the adjacent sides of the column. It

Model	Load case (i)		Load case (ii)	
	conforming	hybrid	conforming	hybrid
Strain energy kNmm	34.992	37.520	77.447	83.630
Bending moment on side 1, kNm	69.91	71.87	0.0	0.0
Torsional moment on side 1, kNm	0.0	0.0	-69.59	-63.42
Shear force on side 1, kN	114.95	119.24	0.0	0.0
Bending moment on side 2, kNm	9.54	9.66	-34.31	-36.69
Torsional moment on side 2, kNm	-25.08	-21.64	35.70	30.84
Shear force on side 2, kN	11.32	10.90	-57.02	-58.48
Bending moment on side 3, kNm	-6.91	-8.75	0.0	0.0
Torsional moment on side 3, kNm	0.0	0.0	-3.83	-3.59
Shear force on side 3, kN	-17.59	-21.03	0.0	0.0

Table 1
Results from conforming and hybrid equilibrium models

should be noted that comparatively large forces occur at the corners of the column, probably reflecting the singularities of the linear elastic theory, the presence of which may also pollute the local solutions. The shear forces from the equilibrium model are piecewise linear by virtue of equilibrating with quadratic moment fields.

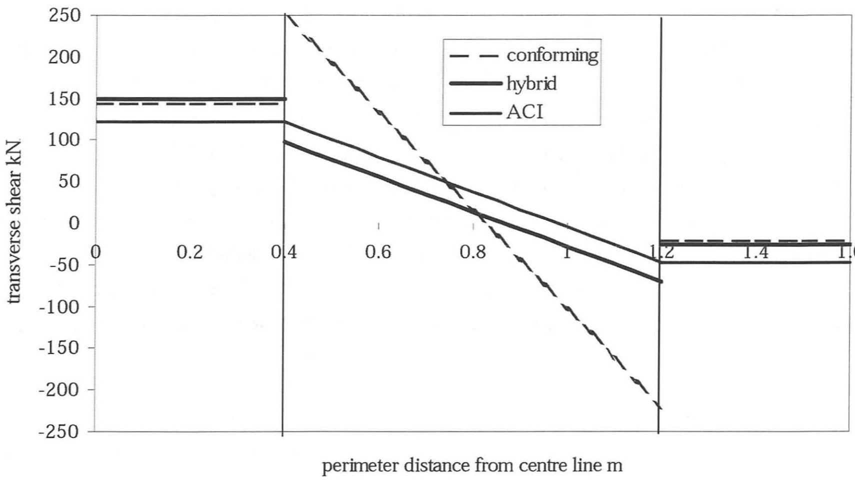


Figure 11
Shear distribution around column perimeter; load case (i)

It is of interest to compare also the distribution in load case (i) with that defined by MacGregor (1997) based on the recommendations of the American Concrete Institute (ACI) for the design of flat slabs. This distribution is labelled ACI in figure 11, where it can be observed that it is in rather better agreement with the hybrid model compared to the conforming one. It should be noted that it is more usual to consider shear distributions around a perimeter distanced from the sides of the column rather than around the column perimeter itself, however practice varies.

Both types of model have lead to a complete description of the load paths into the column, however that from the equilibrium model is formed directly from edge tractions of a model which contains a minimal number of elements, whereas that from the conforming model requires a relatively fine mesh and significant post-processing of results which may be dominated by singularities.

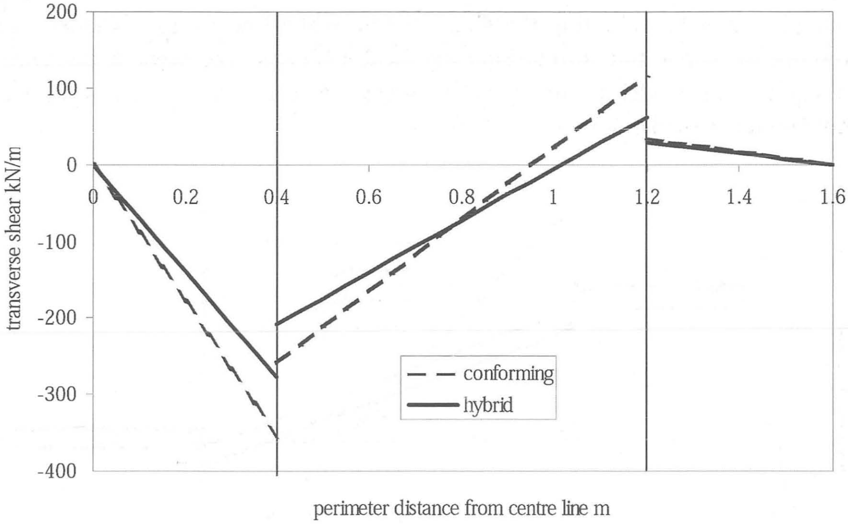


Figure 12
Shear distribution around column perimeter; load case (ii)

Stress trajectories

Another interesting avenue of development relating to equilibrium models concerns the visualisation of stress fields as stress trajectories. These were common in the days when photoelasticity was a popular method of stress analysis (Frocht 1941) although their derivation appears to go back to the pioneering work of Maxwell (1850) as reported by Timoshenko ([1953] 1983). With the advent of computational techniques for stress analysis, the popularity of trajectories appears to have waned, possibly due to inherent numerical difficulties, and to have been replaced by stress contour maps which give an incomplete picture since they only deal with a single component.

However there remains a need for this form of stress visualisation in the fields of mechanical and structural engineering, e. g. to provide guidance in defining force paths as required in the design of reinforced concrete structures using strut and tie concepts (MacGregor 1997) and as a design aid for deciding forms and dimensions in mechanical engineering components (Thamm 2000). This means of visualisation appears to be more natural for equilibrium models since they provide directly the necessary data (Pereira & Almeida 1994).

A simple example illustrates results for a concrete corbel structure where compressive and tensile trajectories are plotted separately in figure 13. In this Figure the trajectories have been colour coded to indicate the magnitudes of the principal stresses thereby presenting a complete picture from which a simplified strut and tie system of forces may be derived.

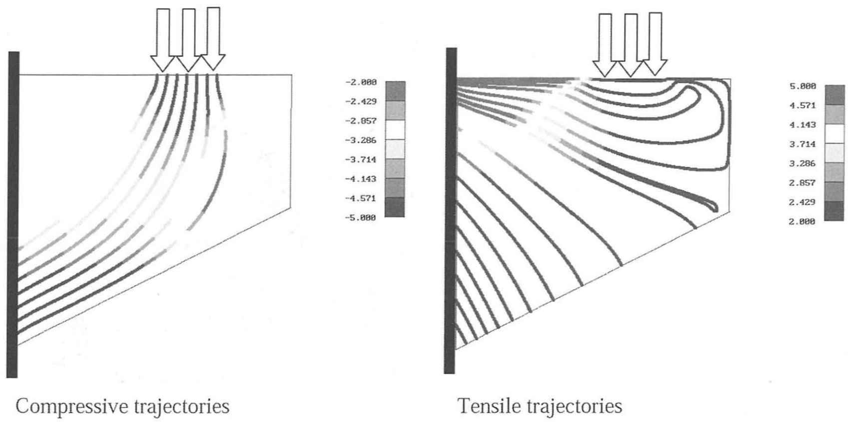
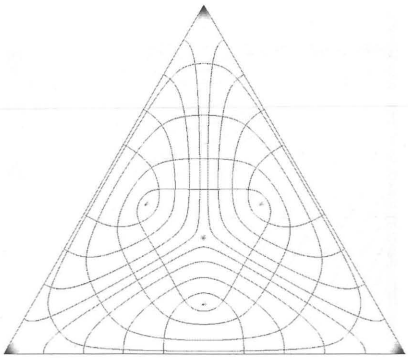


Figure 13
Stress trajectories for a concrete corbel

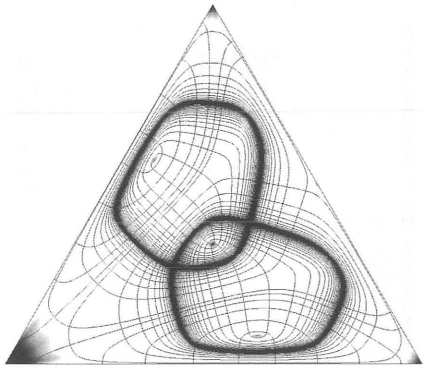
Commercial FEM software does produce approximations to trajectories in the form of Maltese crosses at discrete points, i. e. tangents to orthogonal trajectories scaled in length to indicate magnitudes of principal stresses. However scaling effects often lead to a confused overlapping set of crosses which can be difficult to interpret. Complications also arise in numerical techniques used to produce continuous trajectories from a known stress field, e. g. due to the presence of isotropic points or zones where the directions of principal stresses are indeterminate, and due to ill-conditioning when trajectories come close to merging.

In the context of triangular membrane equilibrium elements it is of interest to compare the trajectories corresponding to the lowest degree self-stressing field with those obtained by Maxwell (1850) in a triangular glass prism. In the case of polynomial fields it transpires that the lowest degree is 4, and then there is a unique mode of stress as shown in figure 14 (a) (Maunder & Moitinho de Almeida 2005). These trajectories appear to be almost identical to those derived by Maxwell from photoelastic measurements on a glass prism which contained a

residual stress field due to heat treatment. It appears that Maxwell derived trajectories from families of isoclinic lines in turn derived from isochromatic curves. From the trajectories Maxwell proposed to calculate principal stress magnitudes by an integration procedure. Such stresses are not reported, but their accuracy would appear to be questionable! This final step is of course the inverse problem to the one of deriving trajectories from known stress fields.



(a) the quartic hyperstatic stress field



(b) a quintic hyperstatic stress field

Figure 14
Stress trajectories for an equilateral hybrid equilibrium element

The trajectories in figure 14 (b) were derived from one of the 5th degree self-stressing fields, and this case illustrates the potential for numerical problems as well as raising questions over the existence of trajectories in the form of spirals as opposed to closed curves. This takes us now into the realm of future research.

Acknowledgements

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Analysis of flat concrete slabs on columns

Christopher T. Morley

On the centenary of the introduction to building practice of the flat reinforced concrete slab supported only on columns, i.e. without beams, this paper gives a brief survey of the development of analytical approaches to such slabs, concentrating mainly on analysis leading to decisions about amount and layout of reinforcement. The first flat slab was the Bovey Building by CAP Turner in Minneapolis in 1906; this was closely followed in Europe by the work of Robert Maillart, who obtained a Swiss patent (Maillart 1909). Since then flat-slab construction has become very widespread, its main advantages being simplicity of required formwork, and reduced overall structural depth (compared with slabs supported by downstand beams). Usually a flat slab is continuous over a rectangular array of columns, with several bays in each direction.

The principal design questions, apart from amount and layout of reinforcement, are control of deflections at the serviceability limit state and, especially, design against shear failure near the (usually monolithic) connections between the slab and its isolated supporting columns. Early flat slabs were commonly built with "drop panels", i. e. local thickening of the slab around the columns; and below that the columns sometimes had spread capitals (Fig. 1 a) —all this intended to tackle the punching-shear problem without thickening the main part of the slab near midspan and thereby increasing the dead weight and bending moments. Maillart usually used extensive curved column capitals, but not separate drop panels. Latterly, exploiting special designs of shear reinforcement around columns, it has been more usual to construct flat slabs with neither drop panels nor column capitals, again to simplify formwork and to achieve full headroom close to the columns.

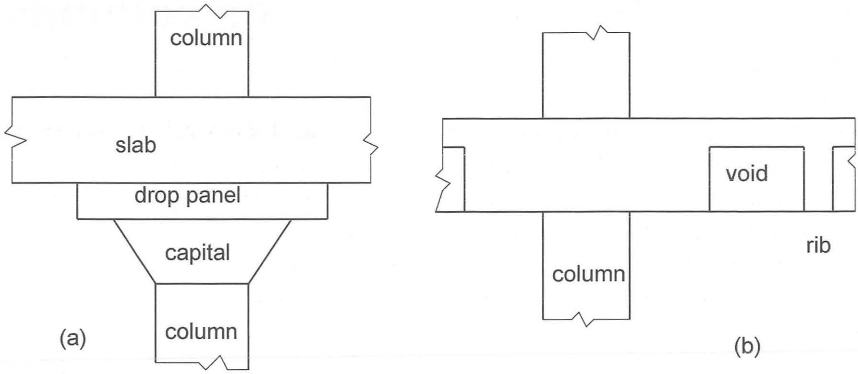

Figure 1
Forms of flat slab construction

A common variant of the flat slab is the "waffle slab", in which formers above the horizontal soffit formwork construct voids, often square in plan, leaving intersecting orthogonal ribs at spacing small compared with the span between columns. Waffle slabs are usually considered as flat slabs of reduced self-weight, with bottom reinforcement concentrated into the ribs. Voids near the column head may be omitted (Fig. 1 b), to assist with the punching-shear problem. Other variants, e. g. broad shallow beams spanning between the columns, on to which the main thin slab spans either one-way or two-ways, are beyond the scope of this paper.

For flat slabs on a rectangular array of columns, design of reinforcement has now been codified into a comparatively routine and well-understood procedure, using either (for slabs satisfying certain limitations) a "simplified" or "direct" design method, or (for more complex cases) an "equivalent frame" approach. However, it took many years for these established procedures to evolve, and some problems remain unresolved for special cases such as irregular column layouts. The aim of this paper is to outline briefly the history of this design evolution, and of the structural analysis and reasoning underlying these modern methods. The paper then goes on to discuss possible analytical approaches to the irregular-column-layout problem. The historical part of the paper relies heavily on three sources, Sozen and Siess 1963, Furst and Marti 1997, and Jones and Morrison 2005.

Current design methods

Current design relies largely on not exceeding certain specified ratios of span to depth, so as to limit deflections under working loads; and selecting adequate concrete strength and depth, often allied with special shear reinforcement, to provide adequate strength in punching shear near columns. Apart from that, the amount and layout of reinforcement against bending moments is based on broad equilibrium considerations, allied to some elastic analysis, taking advantage of some redistribution of moment (typical steel proportions being quite low) —all backed up by experiment and practical experience. Because of the absence of beams, a flat slab has to carry the entire load spanning both ways between the rows of columns.

There seems to be little in the Eurocodes specifically about flat slabs, but the British BS 8110 and the American ACI 318–1995 allow "simplified" or "direct" design respectively for interior rectangular panels satisfying rather similar conditions (at least three continuous spans each way, for ACI panel aspect ratio < 2, successive span lengths not differing by more than one third, imposed load/dead load < 2 (for ACI, BS has 1.25 here)). Design is usually for the maximum uni-

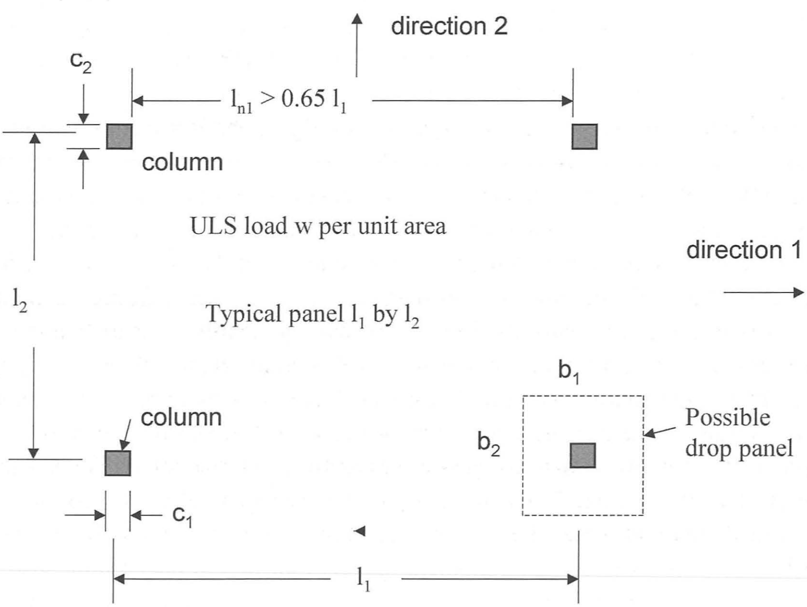

Figure 2
Basic panel dimensions in plan

	Total moment M_0	Percentage of M_0 in hogging	Percentage of hogging moment in column strip	Percentage of sagging moment in column strip
ACI 318 -95	$\frac{wl_2 l_{n1}^2}{8}$	65	75	60
BS 8110	$\frac{wl_2 l_1^2 F}{8}$	50	75	55

$$F = (1 - 2c_1/3l_1)^2$$

Table 1
Coefficients for interior panels

form load applied everywhere, as for these conditions pattern loading is not significant (and membrane or dome action will assist somewhat when a loaded panel is surrounded by unloaded ones). In both codes the reinforcement in one direction (say direction 1 in figure 2) must be designed for a total moment M_0 per panel closely related to the well-known free-span moment WL/8 (but for details see below) . In both codes a certain proportion of M_0 is taken as hogging moment by top steel along the line of columns, the rest by bottom steel at midspan between columns —but the selected proportions differ somewhat (table 1).

In both codes the typical panel is divided into “column strips” and “middle strips” both ways (see figure 3 for direction 1). The column strip is the drop-panel width b_2 if a drop panel of reasonable width is provided, but otherwise the sum of one quarter of the direction 2 widths of panels either side of the column line concerned; and the rest of the panel constitutes the middle strip. A certain proportion of the total hogging moment is then allocated to the column strip, the rest to the middle strip; and a certain proportion of the total sagging moment is likewise allocated to the column strip. These strip proportions are remarkably similar in the two codes (table 1). Reinforcement is then provided for these strip

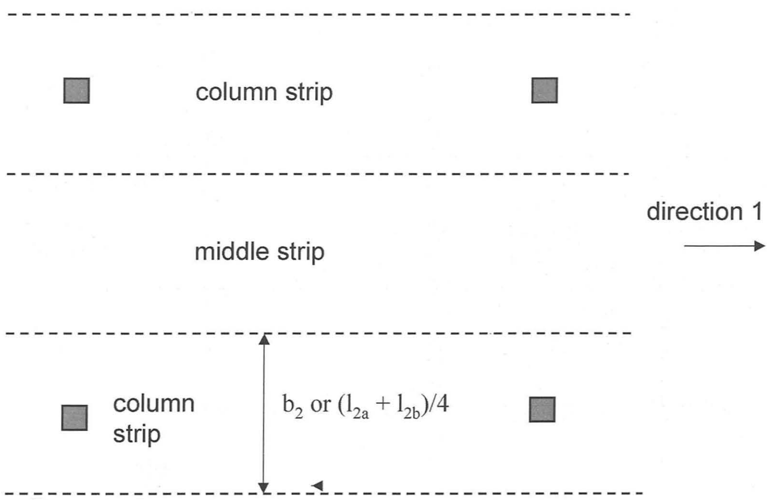

Figure 3
Division into strips, for direction 1

moments, usually evenly spread across the strip though the top steel in the column strip may be concentrated towards the column itself (perhaps with an eye on the elastic distribution).

Two questions arise on these empirical approaches, and are considered below: whether there is any particular rationale for selecting the column strip as essentially half the panel width; and whether there is any rationale for the proportions given in table 1.

For panels not satisfying the above conditions, and for any significant lateral or pattern loading, an "equivalent frame" analysis is suggested, again quite similar in ACI 318 and BS 8110. Recognising that the slab has to carry full load both ways, orthogonal two-dimensional multi-storey rigid-jointed frames "cut" from the structure are analysed, each consisting of a row of columns with associated slab forming the notional beams of the frame (e. g. Fig. 4) and carrying the full loading. The "associated slab" is the sum of half the width of the panels on each side of the column line, though some adjustment to stiffness may be made, since the columns are connected to the real slab at isolated points and not across the full slab width.

Typical frames running through the middle of a building are analysed, as well as frames along the periphery to study loading on edge and corner columns. This

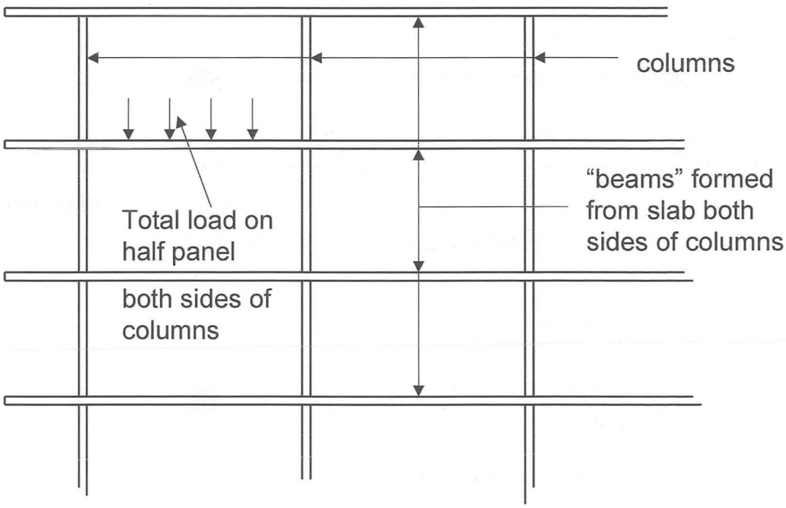

Figure 4
Typical "equivalent frame" across structure

analysis is by moment distribution or equivalent computerised elastic methods, perhaps somewhat simplified to recognise the effects of ductility, all in the spirit of the lower bound theorem of plasticity theory. The aim is twofold, firstly to assess moment transfer to the columns about both axes, for use in column design. Secondly —what primarily concerns us here— the frame analysis produces total moments across the panels in both directions, usually hogging along the lines of columns, sagging at midspan. For reinforcement design, these moments are then assigned to column and middle strips using table 1, so similar questions about this assignment arise with the "equivalent frame" method as with the empirical design approach.

Evolution of the "direct" method

A fascinating account is available (Sozen and Siess 1963) of the uninhibited argument and discussion in the United States, in the years following 1906, about how to analyse and design flat slabs. The total weight of reinforcing steel in the same 6 m square interior panel varied by a factor of 4 across seven different design methods (McMillan 1910), with the originator Turner using the least steel (despite placing it in four directions not two). Turner's 1906 building had performed

satisfactorily in a load test, and innovative strain readings were taken on a flat slab floor under test (Lord 1910). These, and similar readings in other tests, tended to produce comparatively low bending moments, when converted using standard lever-arm theory. However, a simple statical analysis (Nichols 1914), leading to a formula for total moment M_0, gave appreciably higher moment values. Nichols considered a free-body diagram of part of two interior panels to one side of a point column (Fig. 5), bounded by three panel centre-lines and the line of columns, cut from a wide continuous array of equal panels all under the same uniform load. Symmetry arguments show that there can be no twisting moments on any of these boundaries, and no shear forces either (apart from the column reaction). Moments about the line of columns then give

$$M_A + M_B = M_0 = wl_2l_1^2/8 \tag{1}$$

and an appropriate modification to allow approximately for the finite diameter c of the column capital produced the BS 8110 formula in table 1, still in use today.

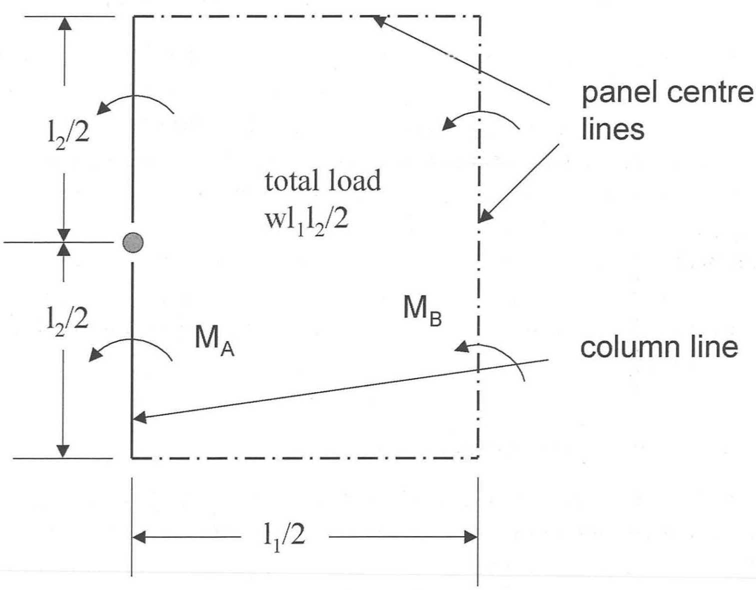

Figure 5
Nichols' free body diagram

Of course, equation 1 covers only the total hogging moment M_A along the line of columns, and the total sagging moment M_B at midspan; it can say nothing about the distribution of moment along those lines, and therefore nothing about the detailed layout of reinforcement. Nevertheless, it was worrying that some of the then-current design methods seemed not even to satisfy (1).

Fascinating detail is given in Sozen and Siess 1963 of how these differences over moment values were slowly resolved, through succeeding versions of the ACI code (one with “even more flagrant violation of statics”), partly by reference to linear elastic analysis (see below), but mainly through realisation that steel strains measured in the tests had been incorrectly converted into bending moments. The test strains had been converted to bar force using Young’s modulus and the bar area, and thence to bending moment by multiplying by the lever-arm of the steel evaluated on elastic theory assuming no tension in the concrete. However, for the gauge lengths and strain magnitudes, and steel proportions, concerned, “tension stiffening” due to bond with the concrete between cracks can have a marked effect (Fig. 6), neglecting which can lead to marked underestimation of the moments.

Nevertheless, it took some time for ACI requirements to converge with Nichols’ statics —in 1925 the ACI code still allowed only 72% of Nichols’ M_0 to be covered by reinforcement , and the ACI formula in Table 1 includes a vestige of the earlier approaches, by using the clear span l_{n1} between internal faces of the

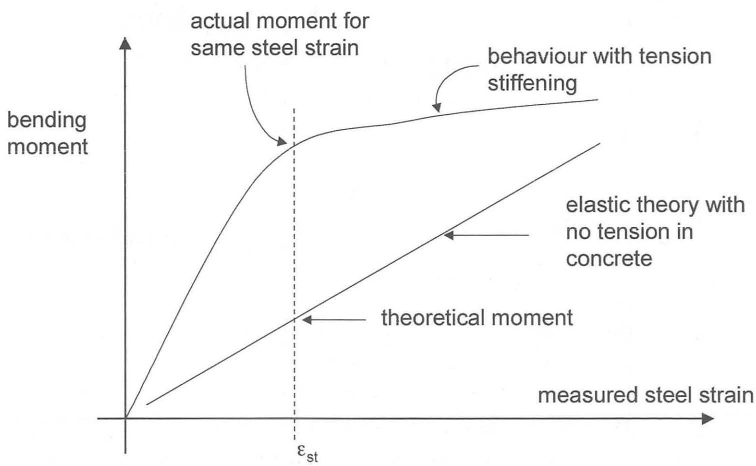

Figure 6
Conversion of measured steel strain to bending moment

columns. The London Building Act of 1930 effectively permitted a reduced design M_0 of WL/10, doubtless relying on membrane action etc, though it did require drop panels at least L/5 square (Jones and Morrison 2005).

Meanwhile somewhat parallel work was proceeding in Europe (Furst and Marti 1997). In 1908 Maillart tested a single-panel and a nine-panel specimen (80 mm thick with 4 m spans), estimating curvatures from the second derivatives of eighth-order polynomials chosen to approximate carefully-measured deflection profiles. These curvatures were converted into moments using the uncracked elastic stiffness of the slab, entirely correct procedure for the load magnitudes and concrete tensile strength involved. However, perhaps because of high susceptibility to experimental error, Maillart's procedures, based on this experimental work, also significantly underestimated the sagging bending moments near panel midspan —compared with linear-elastic analysis (and also with Nichols' equation (1)) (Furst and Marti 1997). Details of Maillart's earliest designs and procedures having been lost, Furst and Marti investigate whether two Maillart designs of 1931 and 1939 nevertheless had adequate reserves of strength —a pertinent question since Maillart explicitly considered sagging moments only, simply bending the midspan reinforcement up towards the columns, to enhance shear strength and provide for hogging moments over the column capitals. Application of a finite-element programme (Steffen 1996), in which various linear-elastic self-equilibrating stress systems are superposed so as to maximise the (plastic theory) lower bound on the collapse load, gave load factors of about 1.3 on the required ultimate design load. Surprisingly perhaps, Maillart could underestimate the midspan bending moments but still arrive at a satisfactory design —presumably due to redistribution of peak moments (exploiting the full ductility assumed in plasticity theory), and perhaps also the extra hogging strength over the columns due to the capitals. So Maillart's pioneering designs are shown to have been fully satisfactory —but further development of flat-slab design in Europe rested on elastic analysis.

Application of elastic thin-plate analysis

Although the bending theory of thin elastic plates had become established well before the end of the nineteenth century, it seems not to have been much applied to flat slabs in early practice. In the United States this all changed with the publication of the comprehensive paper by Westergaard and Slater (1921), expounding the elastic theory and applying it to practical problems, using finite-difference solutions for various load cases, considering the effect of columns and capitals on stiffness and stresses, and comparing theory with test results —all informing the ACI code evolution mentioned above.

In Germany flat slab construction was initially inhibited by the lack of recognised structural theory, and in the late 1920's moments had to be determined by appropriate elastic theory (Lewe 1920 or Marcus 1924). Lewe used Fourier series, taking the support reactions as uniformly distributed over small squares and allowing for Poisson's ratio —his 1922 analysis of alternating strip loading has recently been corrected (Furst and Marti 1997). Marcus used finite differences for several geometry and load conditions.

The modern tendency in linear elastic analysis is to replace the traditional methods of solving the basic plate equations (e. g. Fourier series), which are best suited to simple geometries and boundary conditions, by the use of Finite Element analysis. Software is readily available for computers, and every variety of structural form and loading can in principle be treated. The main difficulty with linear FE analysis (Jones and Morrison 2005) is its tendency to predict high values for the peak hogging moments over columns —a tendency likely to be exacerbated if shear as well as bending deformation is allowed for. It is difficult to get linear FE to agree with the well-established design approaches (table 1) for the simple cases they cover, without allowing some spreading or "averaging" of the peak moments, to recognise the stiffness changes due to cracking, and eventual redistribution of moment due to yield of the steel (which are of course at the root of the simplified design methods). But then how do we know what 'averaging' to do in the non-standard cases, not covered by table 1, when FE analysis is used?

Non-linear Finite Element programmes are indeed under development, which can allow for such phenomena as cracking of the concrete, yield of the reinforcing bars, and bond-slip —and would also (if given the correct boundary conditions) incorporate the development and decay of compressive membrane action as deflection proceeds, as affected by column stiffness etc. However, NLFE is as yet rather specialised and not much deployed in design offices. In addition, it would suffer from perhaps being affected by a range of possible pre-existing cracking, yield or self-equilibrating stress systems, due perhaps to temperature or foundation settlement or previous loading cases —none of which affect the simplified methods of table 1 which are essentially based on equilibrium under a standard loading.

Twisting moments

It is surprising, given their prominence in well-known elastic theory for thin plates, how little mention there is of twisting moments (figure 7, evaluated on the main x, y axes in the steel directions), either in the various design codes or in historical commentaries on flat slab design. True, in the standard case of uniform

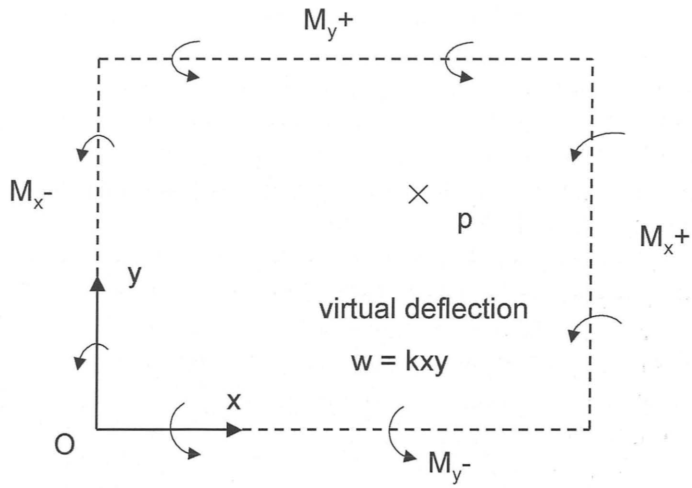

Figure 7
Bending and twisting moments on slab elements

load over many continuous equal panels, there can from symmetry be no twisting moments on the lines of columns or on the panel centre-lines —and bending moments in these positions determine the main amounts of reinforcement required. However, the deflected surface of the slab will certainly involve some twist away from these lines, so that twisting moments should presumably affect the layout of that reinforcement, and in particular the distance it needs to extend into the slab before being cut off. Furthermore, after cracking the stiffness against torsional moments can be appreciably less than that against bending moments aligned with the steel —so that twisting effects should be considered in any discussion of slab deflections beyond first cracking.

Flat slab design practice and Codes had become fairly well established, certainly for rectangular panels, before the effect of twisting moments on the required reinforcement in slab elements was fully realised, by Johansen and by Hillerborg (Hillerborg 1953, Wood 1968). On certain simplifying assumptions, e. g. about lever-arms for the steel bars, the bottom reinforcement in the x-direction should be designed for an equivalent moment per unit width M_x^*, which is the actual ultimate applied bending moment M_x plus a factor k times the twisting moment magnitude $|M_{xy}|$, the steel in the y-direction being augmented by $|M_{xy}|/k$. On varying the factor k the optimum solution, minimising the total steel in the element, emerges as

$$M_x^* = M_x + |M_{xy}| \tag{2}$$

with modified equations required when (2) yields a negative M_x^*. The effect of the modulus sign in equation (2), and the need to consider top and bottom steel in both x- and y-directions, is that a twisting moment acting alone can require four times as much reinforcement as a bending moment of the same magnitude aligned with one set of bars.

It was evidently an appreciation of the diseconomy of twisting moments, as well as its engaging statical simplicity, which led to the development and widespread adoption of Arne Hillerborg's "strip method" for slab design (Hillerborg 1956), applied first to slabs supported by beams, and later to flat slabs on column supports. The ductility of the steel (in the low proportions typical of slabs) is relied upon to permit unlimited redistribution of moment as collapse approaches, so that a designer may arbitrarily set the twisting moment M_{xy} to zero throughout a slab, seek equilibrium load paths requiring only bending moments M_x and M_y, and provide reinforcement accordingly.

The strip method usually considers a series of orthogonal intersecting beams, each of infinitesimal width —but Gurley in 1979 introduced a method (Gurley 1979) of treating a wide rectangular zone (with edges parallel to x- and y-) carrying zero twisting moment throughout. A virtual displacement of the form kxy into a warped ruled surface is then considered, giving no internal virtual work because of the absence of curvature in the x- and y-directions and since M_{xy} is zero. The virtual work equation then shows that the sum of the "bimoments" of the applied loads about the origin (load times *both* x and y) must be balanced by the sum of the bimoments arising from the bending moments along the four zone edges (moment times either x or y as appropriate) (Fig. 8).

The Hillerborg strip method applied to flat slabs

An attempt can be made to apply Hillerborg's strip method to continuous flat slabs as follows, to generate an equilibrium system of moments, considering uniformly-distributed load w and using the standard direct-design division of the panel into middle and column strips each half the width of the panel. (There is a certain difficulty of nomenclature here, in that the word "strip" is used in two different senses —in the direct-design method the middle strip and the column strip are of finite width related to the panel span; in the Hillerborg method as applied here, the strips are of infinitesimal width, and lie within the larger strips of the direct design approach.)

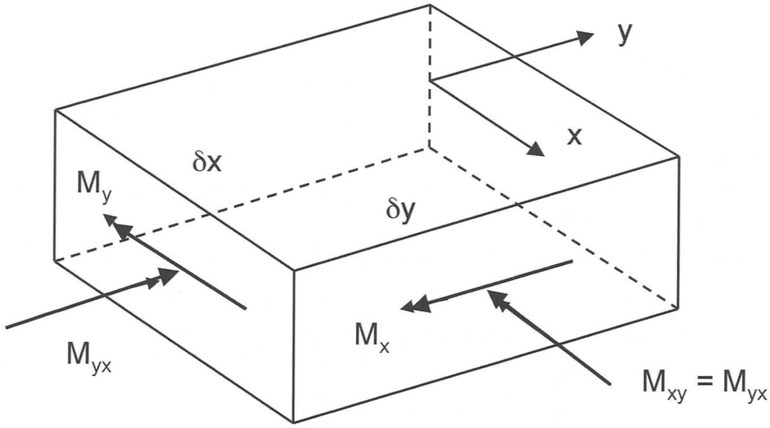

Figure 8
Zone considered for bimoment equation, when $M_{xy} = 0$

Loads applied near the centre of the panel, where the two middle strips intersect, will be carried in unknown proportions, say p_x and p_y in the two directions, summing to w. Each finite-width middle strip is made up of a large number of infinitesimal Hillerborg beam-strips, and to produce a uniform spread of required reinforcement across the middle strip, every Hillerborg strip within it must have the same pattern of loading and of bending moment. Thus p_x will be a constant, and the load, shear and moment diagrams for a typical Hillerborg strip A-A in figure 9 will be as shown in figure 10 (considering only the part from midspan to the line of columns). Similar diagrams apply to the strips BB in the y-direction with load p_y and length l_2.

By symmetry there can be no shear force in this Hillerborg strip at midspan or on the line of columns, and if the Hillerborg strips within the supporting column strip are all to have the same moment diagram, the reactions on to the middle strips must be uniform as shown.

A uniform bending moment could be added to the strip while still preserving equilibrium, and here we have chosen to arrange that the moment magnitudes are equal at midspan and on the line of columns, permitting equal top and bottom reinforcement to be provided. This gives a point of contraflexure at quarter-span, precisely where the reaction from the supporting column strips commences.

It is interesting to note that, if one could vary the bending strength by cutting off the reinforcement precisely according to the moment diagram, and search for

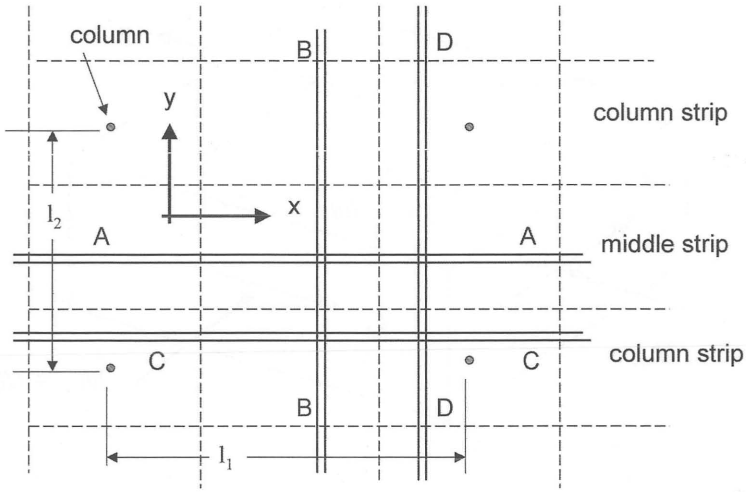

Figure 9
Hillerborg strips for typical rectangular panel, u.d. load

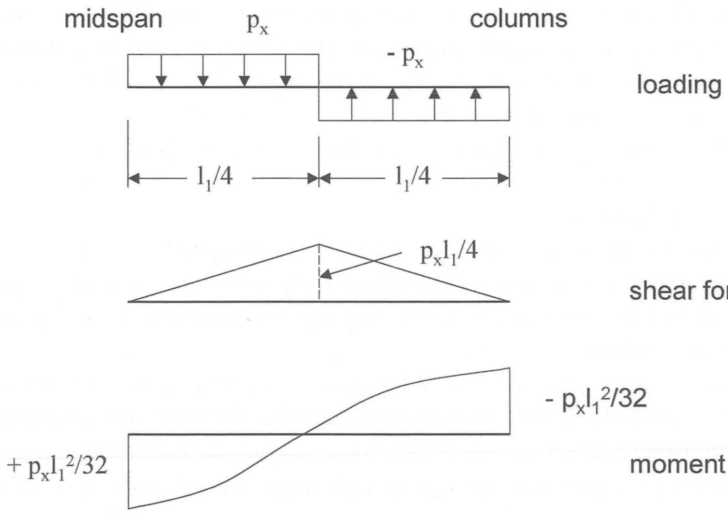

Figure 10
Loading, shear and moment in strip A-A

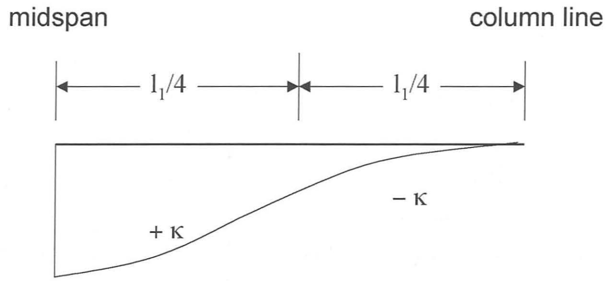

Figure 11
Deflected shape of minimum-reinforcement solution for strip

the solution with minimum total reinforcement (in this isolated strip) Heyman's theorem (Heyman 1959) indicates that this "minimum weight" solution would permit plastic collapse with constant magnitude of curvature throughout the strip. This theorem requires the deflected shape shown in figure 11, with contraflexure at quarter-span —and this can agree with the switch from downward to upward loading, as seems reasonable, only if the direct-design middle strip width is set at half the panel width. So perhaps there is some rationale, apart from simplicity, behind the seemingly arbitrary choice of the middle-strip.

The upward reactions on strips AA (Fig. 10) must be taken as loads on the narrow strips CC within the y-direction column strip, in addition to the applied load w. So the loading, shear force and moment diagrams on strips CC are as in figure 12. Here the point of contraflexure has again been chosen at quarter-span, as would of course be possible.

Sadly, the shear force and moment diagrams between quarter-span and support have to be shown dotted; they are essentially unknowable, and must vary across the column strip, because the support at the right is not provided uniformly across the column strip, but is concentrated at the column. This fact was early recognised as a major limitation upon the application of the Hillerborg strip approach to the flat-slab problem. Isolating the "mushroom" zone where the column strips intersect (Fig. 13), and considering the simplest case with zero moment along the edges, what is needed is a simple equilibrium solution to transfer the edge shear forces and panel loading on to the column.

For columns of finite dimension it may be possible, though laborious, to start at the column with load wl_1l_2 on an assumed rectangular patch, and invent a series of intersecting beams of width related to the column size, taking part of that load and transferring it via uniform reactions over further rectangular zones on to orthogonal beams which carry the load further from the column, until eventually

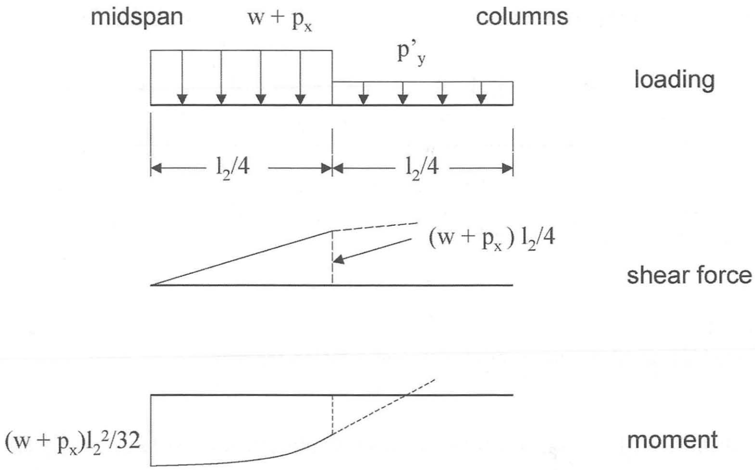

Figure 12
Diagrams for strips CC within column strip

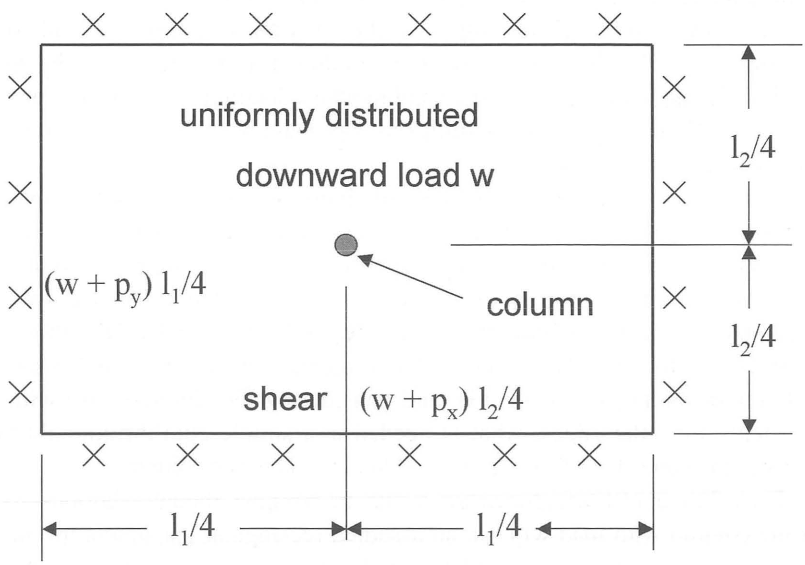

Figure 13
Mushroom zone near column

the whole mushroom zone is covered. A more difficult problem arises with a "point" column, of size small compared with the panel spans.

To tackle such problems Hillerborg himself developed the "advanced strip" method (Hillerborg 1982), in which the assumption of zero M_{xy} is abandoned, an equilibrium field combining a Cartesian field and a field in polar coordinates centred on the column is developed, and reinforcement is provided using equation (2). The resulting moment field has comparatively uniform hogging moment along the line joining the column centres, but equation (2) demands the provision of significant amounts of bottom reinforcement near the column.

More recently, Morley has shown that it is possible to solve the mushroom-zone problem for a point column, while retaining the assumption that M_{xy} is zero throughout (Morley 1986) —on the (perhaps cavalier) assumption that shear strength is always adequate, so that shear failures do not occur, even near the column. The edge shear forces and panel load (Fig. 13) are carried along the Hillerborg strips (prolonging the diagrams in figure 12) until each Hillerborg strip reaches a diagonal of the column zone. At the diagonal there are coincident jumps in the moments M_x and M_y in the strips, which remain constant along the rest of the strip to the next zone diagonal. Examination of the equilibrium of a small element at the diagonal shows that the load is carried as a concentrated shear force along the diagonal to the column (Fig. 14) (suggesting incidentally

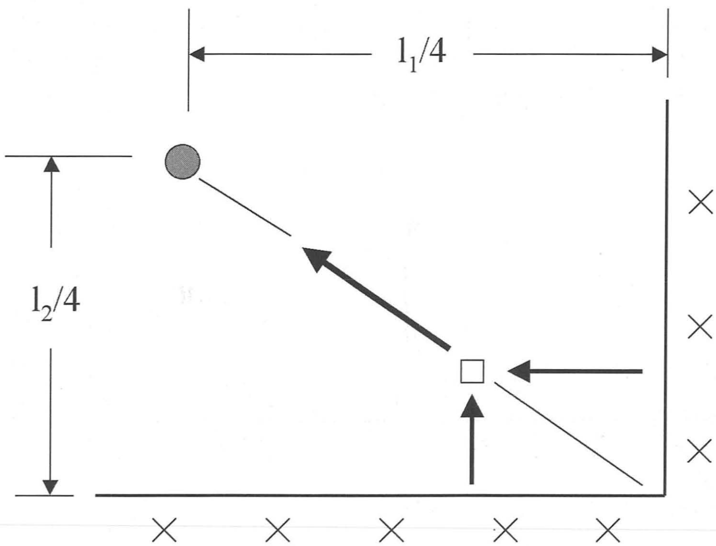

Figure 14
Load transfer along a diagonal

that special shear reinforcement for such slabs should be placed along these diagonals). Only top steel would be required, but the hogging moment magnitudes increase very markedly towards the column, so that this does not give a practical reinforcement layout.

Taking a square panel as an example, table 2 gives coefficients for M_0, for the proportion of M_0 taken in hogging, and for proportions of moment taken in the middle and column strips, on this theory. Comparison with table 1 shows that there is rather more hogging moment than in the direct-design Code methods, and much more moment in the column strip than in the middle strip (and the top reinforcement in the column strip should be concentrated very much over the column head). However, if it were possible to provide variable amounts of bending reinforcement cut-off according to the moment diagram, this layout would use the minimum amount of steel, as it fulfils the Heyman criterion as extended by Morley to slabs (Morley 1966). The optimal solution permits collapse with constant magnitude of curvature at all points, and a corresponding moment field. For slabs with reinforcement permitted in any direction, principal curvatures are considered, but for steel in fixed x, y directions only curvature in those directions is of interest, and twist κ_{xy} can be of any magnitude. The displacement field required for this problem has the profile of figure 11 in both directions, shifted downward as necessary. A somewhat-similar optimal solution for flat slabs on point columns, with piecewise-constant not fully variable steel in the x,y directions, is given by Ahmad and Datta, who conclude that the width of the column strip should be 0.42 times the panel width (c. f. 0.5 times in the direct-design method) (Ahmad and Datta 1989).

	Total moment M_0	Percentage of M_0 in hogging	Percentage of hogging moment in column strip	Percentage of sagging moment in column strip
Hillerborg/ Morley (point column)	$wl^3/8$	75	92	75

Table 2
Further coefficients for interior square panels

Kinematic analysis, yield lines, and plasticity theory

Most of the foregoing has been statical analysis, in several places based tacitly on the lower-bound theorem of plasticity theory, and assuming that the materials exhibit adequate ductility. This is an anachronism for flat slabs, since they were widespread in practice, and design codes for them well developed, before the theorems of plasticity were fully formulated in the 1940's and 1950's (Gvozdev 1938, Drucker, Greenberg and Prager 1952, Hill 1950, Baker, Horne and Heyman 1956).

According to the lower-bound theorem, a loading for which one can find an equilibrium system, satisfying the boundary conditions and everywhere within the yield criterion (or strength) of the materials, will be less than or equal to the collapse load. So in design against collapse under the specified ultimate loading, it is satisfactory to discover any equilibrium system carrying that load, and provide appropriate material strength at every point —since the lower-bound theorem immediately applies to the structure as designed. Such ideas underly much of the foregoing statical analyses.

However, plasticity theory also has an upper-bound theorem, that a postulated collapse mechanism for a given structure, compatible with any boundary conditions on displacement, will give an overestimate of the collapse load if external work done by the loads on this mechanism are equated to the internal energy dissipated. Despite being on the unconservative side, this kinematic approach has also been much applied to flat slabs, particularly in its special form known as yield line theory. This approach also began to be applied, at least to slabs on

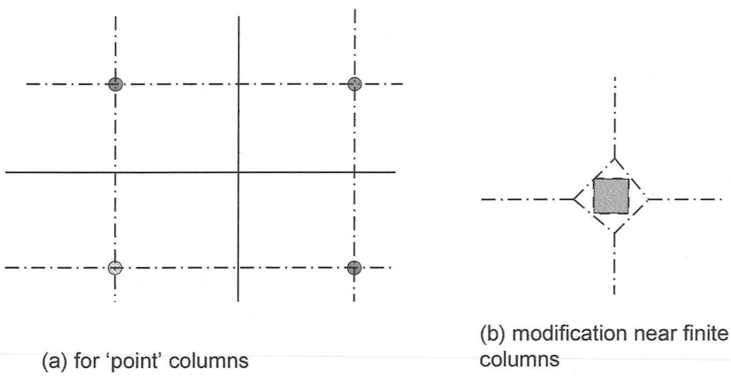

Figure 15
Valley collapse mechanisms

beams, before plasticity theory was fully developed (Ingerslev 1923, Johansen 1943). A special class of collapse mechanisms is considered, in which plastic deformation is confined to a series of straight linear hinges or "yield lines" across the slab surface, with extensive slab regions between the yield lines remaining rigid during collapse. This approach is particularly useful for checking the strength of existing structures.

In applying yield line theory to design flat slabs, it is usual to assume a basic layout of reinforcement, with an extensive mesh of bottom steel especially near midspan, and top steel concentrated near the column heads. Design of the bottom steel is then often controlled by overall valley mechanisms of the kind shown in figure 15 (a), with sagging hinges at midspan and hogging hinges along the column lines, and rigid slab portions rotating about axes along the diagonals. Finite size of column can be allowed for by amending the hogging yield lines near the column heads, as in figure 15(b). Design of the top steel near the columns is often controlled by a mechanism of the kind shown in figure 16. Here most of the slab descends uniformly downwards while a local punching (bending not shear) mechanism occurs at each column head. Of course, it is necessary to consider a wide range of possible mechanisms, and to vary the positions of the yield lines by changing pattern parameters, so as to arrive at the most critical mechanism(s).

A recent investigation (see Jones and Morrison 2005) of several 7.2 m square flat slabs in a six-storey structure at Cardington, each slab designed by a different method, concluded that the most efficient design and construction method was

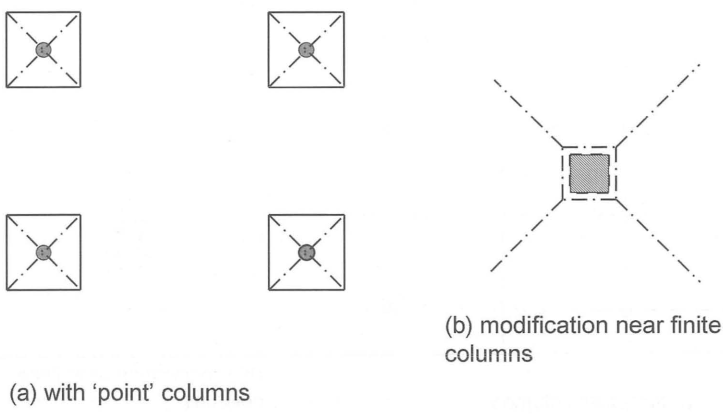

Figure 16
Punching mechanisms at several columns together

the yield-line approach. This is possibly because one can decide to use a practical steel layout, with fairly large regions having uniform reinforcement, before embarking on any calculations —and need not bother to consider bending moment diagrams at all. Various computerised versions of yield line theory are available (e. g. Middleton 1997) and could doubtless be adapted for the flat-slab situation.

Irregularly-spaced columns

Perhaps the most important problem still outstanding in flat-slab design is how to deal with irregularly-spaced columns, when the design codes were mostly developed with a regular rectangular array of columns in mind. Some codes do permit a column to be displaced from its proper position, sometimes by up to one-tenth of the panel span. However, nowadays there is a trend for architects and developers to wish to place columns or wall supports in distinctly irregular patterns.

Given its demonstrated efficiency for rectangular grids, it seems likely that yield line theory will be useful for irregular cases, using compatible collapse mechanisms, perhaps obtained by some suitable distortion of figure 15 or figure 16. Another possible approach would be finite element analysis, but the problem of spreading peak moments over a reasonable width of slab is again encountered. One might think of attempting a strip method of design, but with the lines between columns not meeting at right angles, it is not easy to see how orthogonal reinforcement could be provided to cope with moments in intersecting column strips. Hillerborg's approach to this problem (Hillerborg 1996) is to settle first upon appropriate and practical orthogonal directions for the reinforcement, the same throughout the slab. Lines of zero shear force are then tentatively identified in the midspan areas, and narrow cantilevered beam portions, not much wider than the columns, are then provided, out from the columns in each reinforcement direction to the zero-shear lines. Arrangements are then made, mainly with sagging moments, to carry loads from the midspan areas towards these cantilevers, and reinforcement is provided accordingly.

An alternative approach has been developed, and applied in his US practice, by Saether (Saether 1994). It is based on the "structural membrane" approach, which turns out to have some affinities with the Hillerborg strip method. For an irregular column layout, the first step is to define some non-orthogonal column strips (in the terminology used above). Straight lines are drawn between the centres of reasonably-adjacent columns (Fig. 17); the mid-points of adjacent pairs of these lines are joined, and then the mid-points of pairs of these are joined. The last set of lines define the edges of the column strips, and the slab area is thus divided into regions (1) of sagging moment outside the column strips, regions (2)

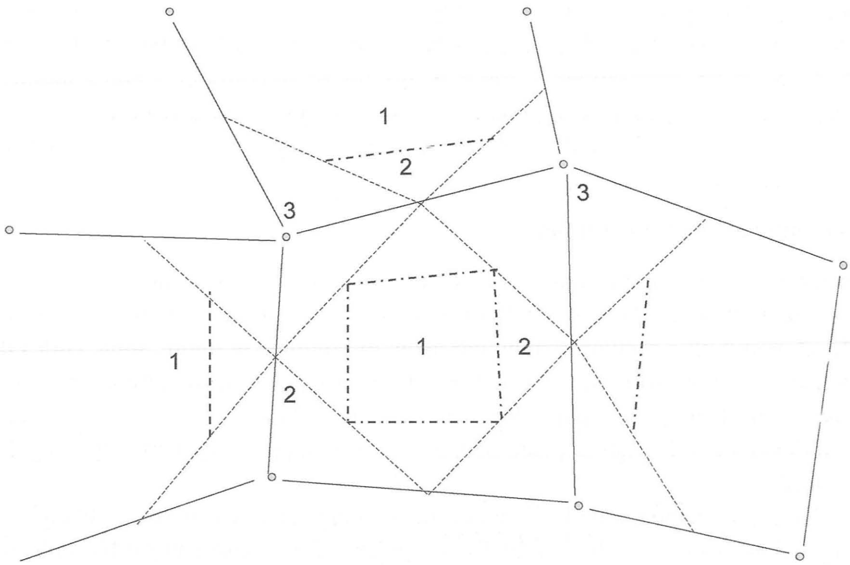

Figure 17
Saether's region layout for irregular column positions

in single column strips with sagging moment along the strip and hogging across, and regions (3) of hogging moment in both directions where column strips intersect over the column heads. One advantage of this approach is that when applied to a regular rectangular array of columns it produces the familiar half-panel-width column and middle strips of the direct design method.

There is then a procedure for determining moment magnitudes at the various points required, based on the rise and thrust necessary for an imagined thin momentless shell to equilibrate the applied loads —the thrust and rise together are nicely equivalent to the moment in Hillerborg strips, for regions of types (1) and (2). For regions of type (2) the hyperbolic shape of Saether's structural membrane corresponds to hogging moments of figure 10 in strips AA of figure 9, and the sagging moments of figure 12 in intersecting orthogonal strips CC. For regions of type (3) near the columns, an approximate moment pattern is developed. In a recent investigation Baskaran has concluded, by studying the out-of-balance bimoments in suitable regions near columns, that in rectangular slabs the torsional moments neglected in the Saether pattern are comparable in magnitude to the torsional moments in the Hillerborg advanced-strip solution —and could be taken into account approximately in steel design using equation (2) (Baskaran 2004).

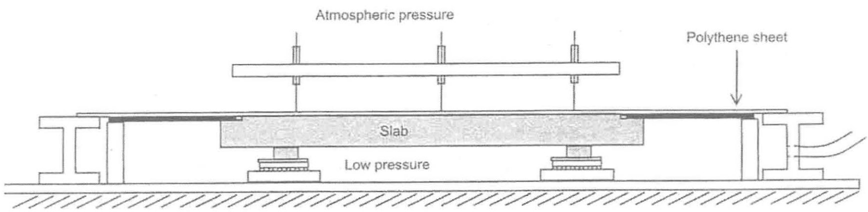

Figure 18
Concept underlying the vacuum rig

Throughout a slab approached by Saether's method, the resulting moments must be transformed into the steel directions; equation (2) applied; and attempts made to smooth out the required steel profile.

As a first step towards testing slabs on fully irregular column grids, Baskaran used Saether's and other methods to design the steel layout in nine flat slabs, uniformly loaded and continuous over many panels, with the columns equally-spaced in a regular array, but on parallel lines giving rhombus-shaped panels with 30 degree skew. Various versions of Saether's method were used, with orthogonal steel directions at various angles to the lines of columns —having one set of bars aligned with a column line seems to be best (Baskaran 2004).

Rather than test a whole floor slab at small scale, typical panels from each design were tested at roughly one-quarter scale, with uniform load applied by vacuum between the test slab and the laboratory floor beneath, in a special test rig (Fig. 18). Continuity with adjacent panels was simulated by having the model slab continue past the columns and out to the likely line of contraflexure in the next panel, and applying shear force on that line using the end reaction on steel plates of appropriate length, also subjected to the vacuum suction (Baskaran and Morley 2004a). Once shear failure near the column heads had been prevented, using special spiral reinforcement designed not to affect the bending strength, the test slabs performed quite well, with most failing in a bending mechanism at close to the design ultimate pressure (Baskaran 2004).

The slabs as built were also analysed using yield line theory, with various mechanisms developed from figures 15 and 16 and adapted to skew panels (see figure 19 in which thin straight lines between column heads indicate hogging yield lines, and sagging yield lines are shown by squiggly lines). Full allowance was made for the yield strength of the individual bars, crossing the yield lines at varied spacing —and prior bond tests had determined what length of bar at the

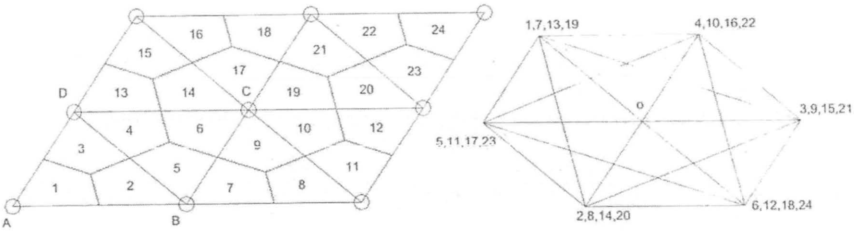

Figure 19
Mechanism with hodograph for a skew column layout

ends should be discounted to allow for bond failure. The collapse mechanism observed in the tests usually corresponded quite well to the most critical mechanism identified in the yield —line analysis, and the experimental failure pressure was fairly well predicted by the yield-line theory (Baskaran and Morley 2004b). However, these tests were of regular, albeit skew, column layouts, and although several other tests have been undertaken, a good deal remains to be done to verify fully the Saether approach to design, and the yield-line method of analysis, for truly irregular flat slabs.

Conclusions

The paper has outlined the century-long history of the design and analysis of flat reinforced concrete slabs on columns, concentrating on the more standard cases of regular rectangular column layout, a wide expanse of slab continuous over many panels, and uniform applied load —and dealing mainly with design for the ultimate limit state, and appropriate layout of reinforcement. After some early disputes, simplified design codes for such cases are now well established, and correspond quite well to the outcome of more detailed approaches such as Hillerborg's strip method.

However, more development work is needed on methods of tackling less standard design cases, such as irregular layout of columns, by such statical approaches as Saether's or by analysis of assumed reinforcement layouts using yield line theory, unless reliance is always to be placed on computerised analysis some kind, and in particular finite-element analysis with some smoothing of moment peaks.

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Practice before theory: The use of the lower bound theorem in structural design from 1850–1950

John Ochsendorf

With the development of plasticity theory, the lower and upper bound theorems have unified elastic analysis and collapse analysis of structures. Both theorems have important implications for structural design, and the ramifications of plasticity theory on design are still unfolding today. In particular, the lower bound theorem is of great importance for design. If the designer can find one set of forces that are everywhere in equilibrium and that do not violate the yield conditions for the material, then the structure is safe and will stand. In the case of hyperstatic structures, there are infinite possible stress conditions and the exact stress state of the structure is impossible to know, since it depends on constantly changing support conditions and states of self-stress. This problem consumed the science of structures for over a century in the search for the "true state" of the structure. However, designers knowledgeable and comfortable with the lower bound theorem do not concern themselves with this "true state". Instead, they simply identify one possible equilibrium state for the structure. If the designer can find a "safe" condition, then the structure can find a safe condition and the actual stress state will not necessarily match the condition postulated by the designer. Heyman (1998) has provided an overview of the historical development of the lower bound theorem.

Consider the case of a fixed-end beam spanning a length L under a uniform load w, which may occur in a framed structure. (Fig. 1). By making assumptions about the support conditions and material properties, Navier and other leading elasticians provided the brilliant insights necessary to arrive at the *exact* answer for a beam of constant section, EI, which has fixed-end moment reactions of

$wL^2/12$ and a moment at the center of $wL^2/24$. (Fig. 1 a) However, this elastic solution is highly sensitive to very small changes in the boundary conditions and is therefore unlikely to be observed in a real structure. An alternative solution is to imagine that the supports do not carry any of the bending moment and that the structure acts as a simply supported beam with all of the bending carried at the center (Fig. 1 b). Another solution is to imagine that the structure acts as two cantilevers with all of the bending moment is carried at each support, as in figure 1 c. Finally, a designer wishing to minimise material use may choose to equalise the moments with $wL^2/16$ at the center and over the supports, which corresponds to the plastic solution, as in figure 1 d. According to the lower bound theorem, also called the "safe theorem", all of these possible solutions are valid design solutions for this problem. As long as the designer identifies one possible equilibrium solution, and ensures that the material is capable of supporting the internal forces without instability, then the structure will stand. Most structural designers in 1850 were not yet confined by Navier's "straitjacket" and were free to postulate possible equilibrium solutions for complex structures (Heyman 1998, 13).

This paper aims to show that many leading structural designers used the lower bound theorem intuitively long before the development of plasticity theory. In short, many structural designers worked only with the equations of equilibrium, which were viewed by some contemporaries as a primitive design procedure, but their methods can be seen as a simple application of the lower bound theorem. The development of elasticity theory allowed for much more complex analysis, and many engineers in the 20th century pursued detailed stress analysis in the hope of determining the one true state of the structure. This paper will show that many

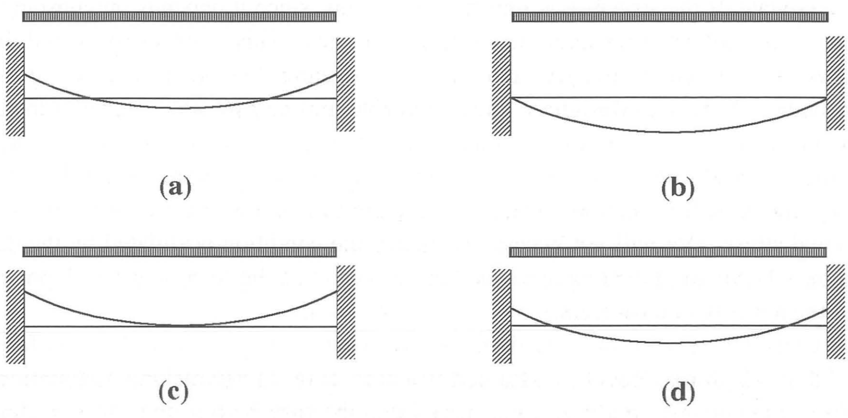

Figure 1
Hypotheses for the possible moment diagrams of a fixed-end beam

leaders in the field consciously rejected more complex elastic analysis methods and instead depended heavily on simpler methods of equilibrium analysis.

To illustrate this, it is useful to briefly consider the historical methods of calculation for various structural types, including suspension bridges, arches, slabs, and thin shells. This survey does not attempt to provide an exhaustive review of all designers or all structural types from 1850–1950. Furthermore, these dates are only meant as an approximate boundary on the examples given in this paper and do not serve to mark the exact dates of major developments. The aim is to illustrate a wide range of structural problems and to demonstrate that some designers were relying heavily on the lower bound theorem, despite the fact that the theorem would not be fully developed until the middle of the 20th century.

John A. Roebling and suspension bridge design

In the static design of suspension bridges, the dominant historical problem was the analysis of the load-carrying role of the stiffening truss in relation to the suspension cable. Buonopane and Billington (1993) have provided an excellent overview of suspension bridge theory from 1823 to 1940. The *elastic theory* of suspension bridges was initially used in the late 19th century to produce bridges with unnecessarily large stiffening trusses, until the development of the *deflection theory* of suspension bridges by Melan (1906), which more accurately considered the inherent stiffness of the cables under dead load. Though these theories did not account for the dynamic loading of suspension bridges, they were safely used for the static design of many suspension bridges in the 19th and 20th centuries.

When John A. Roebling began designing suspension structures in the 1840's, he developed his own design methods. For his first structure, a suspended aqueduct over the Allegheny River in Pittsburgh, Roebling designed a wooden trussed element to carry the water. His calculations for the system are described as follows:

> The original idea upon which the plan has been perfected, was to form a wooden trunk, strong enough to support its own weight, and stiff enough for an aqueduct or bridge, and to combine this structure with wire cables of a sufficient strength to bear safely the great weight of water ("Wire Suspension" 1845, 309).

In other words, Roebling assumed that the wooden structure was sufficient to carry its own self-weight as a beam, and that the suspension cables would carry the water. Of course in reality, the distribution of loads would be more complex than this, but Roebling was confident that such a scheme would work.

As he gained experience in suspension bridge design, Roebling developed more complex systems with higher degrees of indeterminacy. By adding diagonal stays to the Cincinnati Bridge and the Brooklyn Bridge, Roebling aimed to stiffen the bridge against wind vibrations. (Fig. 2) The diagonal stays add an additional load-carrying system, as if the bridge were composed of both a suspension system and a cable-stayed system. Though further research is needed on Roebling's design methods in detail, it is clear that he proportioned both systems to carry load and called for the diagonal cables to be tightened, or "tuned" so that they carried some of the load. In his report on the Brooklyn Bridge, Roebling states that the load capacity of the stay-cable system is so great that the bridge would still stand even without the main suspension cables (Roebling 1867, 24).

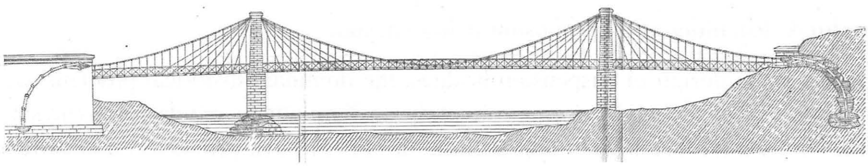

Figure 2
Brooklyn Bridge (1883) by John A. Roebling

How could Roebling have designed such a highly indeterminate system? The structural analysis would be difficult today with non-linear finite element programs and the exact state of the bridge would be nearly impossible to predict. A very small change in assumptions or support conditions, such as the length of the vertical suspenders, would give drastically different results for the internal force distribution. Yet Roebling had the confidence to predict that the bridge would last "indefinitely" if well-maintained.

For all of his designs, Roebling applied simple equilibrium calculations, which proposed one possible force distribution for a given loading. The fact that the actual forces in the structure were almost certainly different from his proposed distribution did not concern him. In his reports, Roebling specified clearly the need for ductility in the wire and this would have given him greater confidence in the use of his simplified design methods (Roebling 1867). He also explored the influence of temperature and showed that the forces in the cables would vary by several tons in different temperature conditions. Through a simplified version of limit analysis, Roebling determined the maximum possible load that would act on each element, such as the condition of the stay cables in the event of the main cable failing, and he designed the elements to have sufficient

strength for the limiting cases. Such a design method would be acceptable today according to the lower bound theorem.

Roebling recognised that despite small unknowable changes in the exact internal forces, the bridge would stand based on his assumed load distribution between the diagonal stays and the parabolic suspension cable. In the decades since, engineers have maintained the bridge by replacing cables when required and by adjusting the diagonal and vertical cables to ensure that any one cable is not overloaded. At least twice in history, cables have failed on the bridge. In 1901, twelve vertical suspenders seem to have broken, and in 1981, two diagonal stays broke (*New York Times* 1901 and 1981). It seems that both failures may have been due to cable corrosion as a lack of maintenance, though it is possible that the cables may have failed due to excessive internal forces. Provided that the cables have sufficient ductility, this should not occur so it seems likely that corrosion caused the failures.

As time went on, other designers did not see the need for the stay cables, though Roebling included them primarily as a means of stiffening his bridge against live loading such as wind. After several years of renovation, the bridge reopened in 1954, with a contemporary engineer's comments:

> As an engineering design the Brooklyn Bridge is outmoded. Suspension systems are simpler today. The diagonal stays would certainly go by the board if the bridge were drastically rebuilt, and there would be fewer and stronger suspenders (*New York Times* 1954).

The drive to reduce the complexity of the structure could in part be attributed to the fact that engineers by 1950 had come to rely heavily on detailed elastic analyses, which sought the one true state of the structure. For the analyst of the 20th century, additional inclined cables only made the calculations more difficult, though they were an essential part of Roebling's designs.

Robert Maillart and arch bridge design

Heyman (1998) has provided an overview of the history of the theory of masonry arches. The lower bound theorem was widely used in assessment and design, beginning with Poleni in the 18th century. In short, designers sought one possible thrust line under the given loading. As long as the thrust line lay somewhere within the masonry, then the arch would stand. This traditional method of arch design was common in the 19th century and early 20th century, and many masonry arches from 100 years ago were designed using the concept of an internal thrust line (Huerta 2004). Of course this is a lower bound approach, which recognises

that the actual thrust line in the masonry may vary within the thickness of the arch. This traditional method came to be regarded as primitive when designers began to seek the "true" state of the arch using elastic analysis. In this context, it is interesting to consider the work of two engineers in the first half of the twentieth century, George F. Swain in the United States and Robert Maillart in Switzerland.

Swain was a professor of civil engineering at MIT and Harvard in the early 20th century. His textbooks on structural engineering grew directly from his teaching and practice. In his 1927 text *Structural Engineering: Stresses, Graphical Statics, and Masonry* he comes close to formulating the upper bound and lower bound theorems in his discussion of arches (Swain 1927, 408–411). He states the lower bound theorem in one sentence: "The arch will stand up if it can". Swain goes on to make a scathing attack on elastic analysis, writing: "to apply the elastic theory even to the reinforced concrete arch is illusory, and a vain seeking after exactness where exactness is impossible" (Swain 1927, 423). Swain's criticism is based on his "long experience" working in practice, and his claim that almost all of the assumptions for the elastic theory are "untrue". He writes:

> The elastic theory is often termed "exact". The assumptions made in it are the following:
>
> 1. That the ends are rigid and do not rotate (this is untrue).
> 2. That the span does not change at all (this is untrue).
> 3. That the material is homogeneous (this is untrue).
> 4. That the modulus of elasticity is constant, not changing with the pressure (this is untrue, though perhaps close).
> 5. That the terms with r in the denominator may be neglected (this may be far from true).
> 6. That the integrals may be replaced by summations (this is approximate).
> 7. That the formulae for flexure are exact (this is untrue).
> 8. The stresses due to shrinkage are neglected.
> 9. That the section is a rectangle (this is untrue).
> 10. That the loads may be determined accurately (this is untrue) (Swain 1927, 425).

Based on the above objections, Swain advocated the use of simple equilibrium methods over idealized elastic calculations.

Similarly, Swiss engineer Robert Maillart rejected detailed stress calculations and recognised the impossibility of exact analysis for complex indeterminate structures. Instead, Maillart developed ingenious equilibrium methods of analysis for the difficult problem of a deck-stiffened arch. As in the suspension bridge,

where the interaction between the cables and the stiffening truss are unknown, the same is true of the deck-stiffened arch. As Billington has illustrated, Maillart postulated that the stiff deck carried all of the bending due to asymmetrical live loading in his deck-stiffened arches and that the thin arch played no role (Billington, 1979, 104). This allowed Maillart to reduce the thinness of his arches well beyond the works of any other designers of the period. In reality, the deck and the arch act together in a complex and unknowable way which depends on the construction methods, support conditions, etc. The structure is statically indeterminate but Maillart was not concerned by this fact and he used lower bound equilibrium conditions to justify the safety of his designs. Many of Maillart's contemporaries were disturbed by his simplifying assumptions and they sought exact solutions to determine the role of the deck and arch combined. In particular, Maillart's competitor Max Ritter and his disciples produced dozens of pages of calculations in their unsuccessful efforts to determine the "true state" of the stresses in the structure (Billington 1997, 178).

Maillart produced fewer than four pages of calculations for his Valtschielbach Bridge of 1925 (Billington 1997, 122). Of these calculations, only half of one page was devoted to the behavior under asymmetrical loads, in which Maillart assumed that the stiff deck carried all of the bending. Billington has demonstrated numerically that this is approximately true (Billington 1979, 104). Maillart used load tests on his completed bridges to verify that the structural forms were behaving according to his simplified assumptions and he gained confidence with

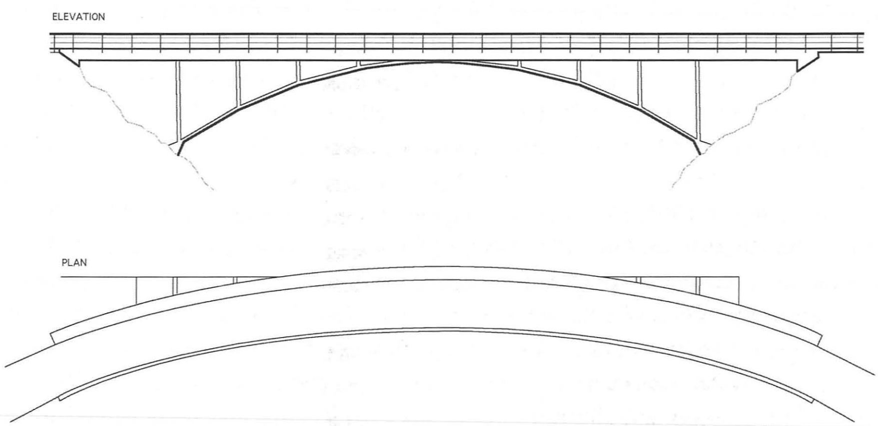

Figure 3
Deck-stiffened arch of Schwandbach Bridge (1933) (after Billington 1979)

each new structure that he built. For the Schwandbach Bridge of figure 3, Maillart again produced very few calculations, using graphic statics to find the form of the arch.

Maillart's calculations are one of the clearest examples of the use of the lower bound theorem. By postulating an equilibrium solution, Maillart was not merely adopting a simplified design method. Instead, he was actively rejecting lengthy and useless elastic calculations that sought the *exact* solution for the internal stresses. In some cases, Maillart used three-hinged arches to reduce the indeterminacy when it was necessary to allow for support movements, as in the Salginatobel Bridge. However, he did not avoid static indeterminacy and was confident in predicting possible equilibrium solutions for very complex three-dimensional structures. Particularly in the case of Maillart, the confident use of the lower bound theorem deserves further study from historians of structural theory.

C. A. P. Turner and the Concrete Slab

Maillart also developed striking new forms in concrete flat slab construction between 1908 and 1910 (Billington 1979, 51–61). Again, rather than derive complex elastic formulae for design, Maillart carried out load tests and measured the actual behavior of the structure. Then he derived simple equilibrium methods for design, which depended on a "mushroom" capital to carry the forces down into the column. The actual flow of forces in a slab with a complex capital is a difficult problem in three-dimensional stress analysis, but for design it is not necessary to resolve it exactly. There are infinite patterns of forces in equilibrium which could be used for the safe design of such a structure. Similar to today's lower bound methods of *strut-and-tie* or *truss* analysis of reinforced concrete, Maillart postulated one possible equilibrium condition and verified his strength calculations with load tests. His reinforcement pattern of parallel bars, with the greatest reinforcing over the columns, continues to be the standard today.

Beginning in 1905, an American engineer C. A. P. Turner developed and built a large number of concrete flat slab buildings a few years earlier than Maillart. Gasparini (2002) has provided an excellent overview of Turner's developments, his design methods, and the controversy that arose from his designs. C. A. P. Turner recognised the danger of punching shear for flat slab construction, and he designed a heavy system of shear reinforcing over the columns (Fig. 4). This allowed him to minimise the reinforcing steel for flexure and he designed for a maximum moment of only $WL/50$, where W represents the total load on the slab and L is the nominal dimension of the slab, allowing him to reduce the amount of steel drastically. It would be interesting to compare the amounts of steel used in

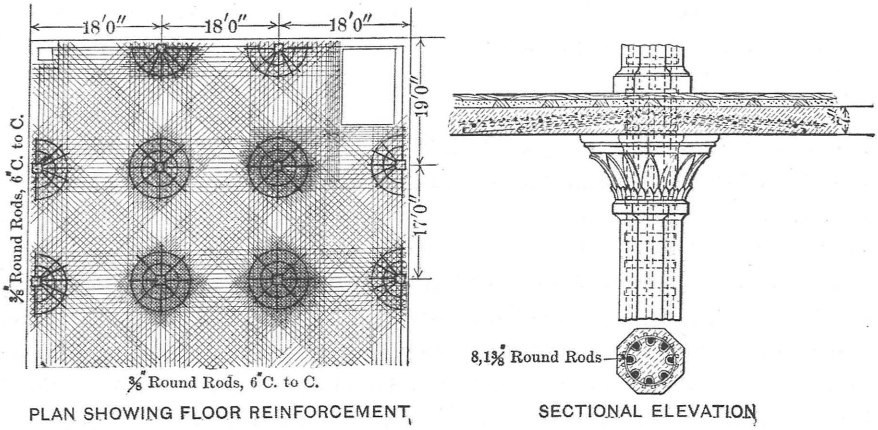

FIG. 96.—The "Mushroom" System.

Figure 4
Mushroom slab system by C. A. P. Turner

various flat slab systems around 1910, in particular the systems by Turner and Maillart, though it is beyond the scope of the current paper.

How did Turner arrive at the coefficient of 1/50? Unfortunately, he did not publish a derivation in his writings, though we can postulate how he arrived at such a number. First, it is clear that Turner had accumulated a great amount of experimental data to relate the strength of slabs to the quantity of steel reinforcing (Gasparini 2002). In addition, it seems that he considered various possible loading paths within the slab, as illustrated by his reinforcing diagram of figure 4. A load applied at the center of a slab could be carried in all directions, though Turner's reinforcing suggests that he envisioned most of the load going directly to the columns. Assuming that the supports are continuous and the moments are optimised to *WL*/16 (as in figure 1), then the moments could be reduced to *WL*/32 to account for the two-way action of the slab. Finally, the moments could be reduced further to allow for load distribution in other directions. Because Turner did not specifically define the patterns of forces in equilibrium, his method is not exactly a method of lower bound analysis. Yet it is quite possible that he arrived at the coefficient of 1/50 through imagining a combined pattern of bending forces in different directions within the structure.

Further research is needed on the analysis and design methods of early flat slab construction. While some in the profession were obsessed with finding elastic solutions for flat plates, leading builders like Turner and Maillart were developing design methods empirically and verifying their methods with load

testing. Felix Candela, another master of reinforced concrete, paraphrased another engineer in his discussion of flat slabs:

> In structures of reinforced concrete, the problem does not necessarily need to be treated by the theorem of elasticity. The only important value from this calculation is that it describes a system in equilibrium, but this equilibrium can be obtained in many ways (Candela 1985, 79).

The wide range of reinforcing schemes used by the designers of early flat slabs illustrates that ideas varied about the actual behavior of the slabs, but as long as the designer provided sufficient reinforcing, the loads could travel according to the reinforcing patterns. This is reminiscent of contemporary work by Rafael Guastavino, Jr. who placed iron bars in masonry structures to provide ties where necessary, but who recognized that there were many possible positions for such reinforcement. For a final example of the lower bound theorem in the 19th and 20th century, it is of interest to consider the design and analysis of thin shells.

Guastavino, Candela, and thin shells

Each of the previous structural types began with simple equilibrium analysis methods, which came under suspicion due to the increasing use of "exact" methods of elastic analysis for indeterminate structures. By contrast, the analysis of shells and domes was almost exclusively a problem of equilibrium analysis up until the middle of the 20th century. Billington (1982) has provided an overview of the development of thin shell theory in the 19th and 20th centuries. In particular, the *membrane theory* is of interest because it is a lower bound method relying only on equilibrium. Rafael Guastavino, Jr. and Félix Candela were two leading figures in the application of membrane theory to the construction of thin shells.

The Guastavino father and son were great builders in the United States in the late 19th and early 20th centuries, constructing large tile domes using the Mediterranean method of timbrel vaulting (Collins 1968). In his groundbreaking study on the history and mechanics of timbrel vaults, Huerta (2003) describes the innovative applications of the membrane theory by Guastavino Jr. in a period before the rise of thin concrete shells. For example, Guastavino postulated that his spherical tile domes acted in compression from the crown down to 52 degrees, as predicted by the membrane theory, and then he provided steel reinforcing or buttressing near the base of the dome. In some cases, Guastavino altered the geometry of the dome to match the thrust line emerging from the upper portion. In all cases, Guastavino applied equilibrium analysis to illustrate that the dome would stand. He was one of the first to apply such analyses to real structures and he was also

willing to depart from accepted geometries where necessary. He was strongly influenced by the method of graphical analysis for domes developed by Eddy in the 19th century and reproduced by Wolfe (1921) and others (Fig. 5). Guastavino's design approach is marked by the confidence of a great builder who knew that if he could provide a path for the forces then the structure would stand. Leading designers in reinforced concrete shells followed the same approach using the membrane theory.

Beginning with the pioneering German work in the 1920's, thin concrete shells grew in popularity and reached a peak in the 1950's and 1960's. While many researchers focused on the infinite complexities of the mathematical analysis of thin shells, leading builders such as Félix Candela used simplified formulae to find equilibrium solutions for complex forms. Candela derived remarkable new forms from combinations of hyperbolic parabolas and he used equilibrium

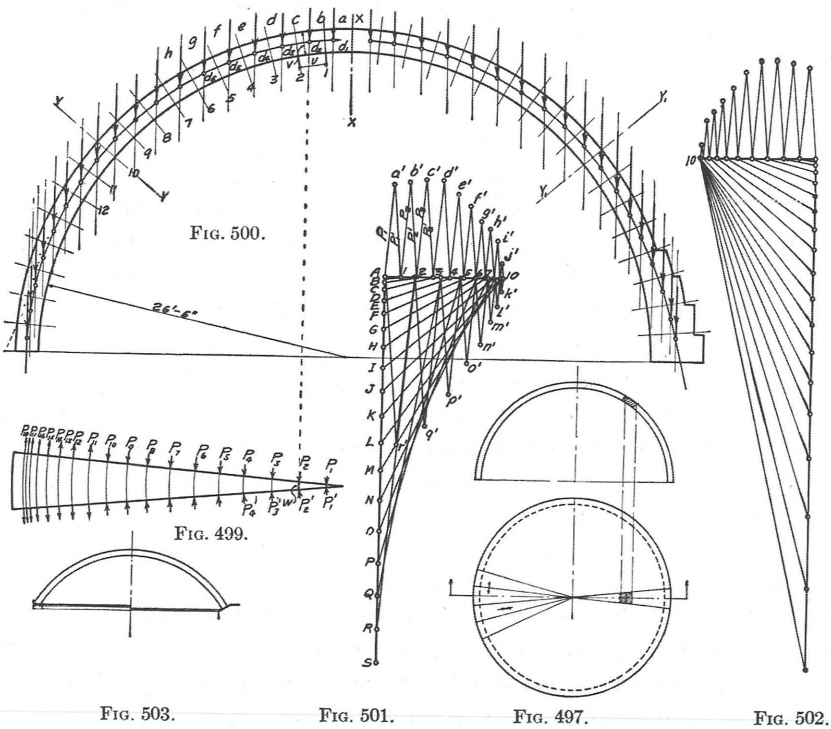

Figure 5
Wolfe's graphical analysis of a masonry dome (Wolfe 1921)

calculations via the membrane theory to demonstrate their safety. Of course numerous other practitioners in thin shells such as Torroja and Maillart also used equilibrium calculations and load testing to verify the safety and performance of shells. Why focus on Candela? He dedicated himself almost exclusively to the design and construction of thin shells and he rose to international fame in the 1950's for his work. His work and his writings illustrate the state of affairs in the mid-20th century and reflect on the future of the profession.

Candela delivered a paper titled "Towards a new philosophy of structures" in 1951 which summarized the current state of structural analysis and made a plea for a return to equilibrium calculations (Candela 1985, 73–90). Candela begins by stating that it is difficult to "demand exactness from calculations, when the problem never has one unique and exact solution" (Candela 1985, 75). This is in direct contrast to many theorists at the time who aimed to work out 8th order differential equations in order to precisely solve for the stress conditions in various thin shell geometries. Candela goes on to describe many objections to the use of elastic analysis and his strongest argument is that it says nothing about the most crucial question: the collapse state of the structure. Echoing Swain's earlier voice, Candela states "that methods based on the theory of elasticity are not appropriate for the analysis of hyperstatic structures of reinforced concrete" (Candela 1985, 83). A later passage is worth quoting in full:

> For many years the exclusive dominance of elastic theories, with all of their mathematical complexity, have made it difficult to do away with the conviction that they constitute the only method to resolve the problem. The gravest sin of the Theory of Elasticity is that its pretension to obtain exact and unique solutions has impeded the search for other perceptions of the problem. It is as if, when looking upon a building or a sculpture, we fix ourselves with only one limited viewpoint, which does not allow us to see all of the characteristics of the observed object. Undoubtedly this would give us a restricted view, and probably erroneous, of the object's true dimensions and conditions (Candela 1985, 87).

Conclusions

The preceding examples are only a few of the many hundreds of possible examples to illustrate the use of the lower bound theorem from 1850 to 1950. The same discussion could be applied throughout the field of structural design including the development of tall buildings, bridge trusses, steel frames, etc. though it would require much more research. The scope of the paper is limited to a few important practitioners who illustrate the historical debate on the merits of simplified analysis. In fact, their contemporaries were also producing lower

bound analyses in the form of elastic stress calculations, but often did not realize that such detailed calculations were unnecessary for indeterminate structures. Each of the designers discussed in the current paper used equilibrium methods as their primary approach to structural design, but more importantly, they rejected uselessly complex stress calculations. Because each of these designers worked as a builder, their primary interest was in producing a safe and functional structure, rather than a detailed stress analysis. Hardy Cross wrote that "what we want is a structure, not merely an analysis" (Cross and Morgan 1950, 1).

Beginning in the 1930's, the development of limit analysis led to a new emphasis on collapse analysis rather than a detailed calculation of working stresses (Heyman 1998). By 1948, one writer declared, "I personally believe that the *overemphasis* on elastic analysis of the last 50 years was an aberration that is losing its preponderance at present" (Van der Broeck 1948). Then the computer changed everything. The development of finite element methods in the last half of the 20th century allowed for the instant solution of nearly any problem in elasticity. At last, the one true state of a structure could be found. And now that finite element analysis can solve any problem in elasticity, a new series of questions have emerged. Foremost among them: how do designers know what structural form to enter into the computer to begin with? Static equilibrium is the starting point and there is much to learn from studying the works of master builders in the past.

Epilogue

During my own undergraduate structural engineering education at Cornell University from 1992–1996, I was taught to approach indeterminate structures exclusively with elastic analysis. There was no mention that the stresses predicted by elastic analyses could not be observed in reality. There was little or no discussion of the collapse analysis of even a simple beam. While I am extremely grateful for the education that I received, I consider this to be further evidence of the continued dominance of elastic analysis and its seductively elegant mathematical formulations.

Acknowledgments

I would like to thank Jacques Heyman, who steadfastly refused to supervise my PhD, but kindly met with me each week for three years to discuss my research. I am grateful for his guidance as well as his continued willingness to meet with me. I would like to acknowledge Santiago Huerta, who pointed me towards many of the sources and ideas in this paper, as well as Chris Calladine, who I turn to with mechanics problems that are beyond my abilities.

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Theory, industry and practice: Carnegie's Pocket Companion and the development of skyscraper construction in Chicago

Tom F. Peters

If theory does not inform practice, it is irrelevant and serves no purpose in building. Irrelevance is a complaint often leveled against architectural theoreticians, particularly in recent decades, but the lack of a connection between theory and practice has rarely been one of structural theory's weaknesses. The one exception I know of is Robert Maillart's invention of the "center of shear". Perhaps there are others, but they are rare. This rarity may be due to the fact that structural theory concerns the physical behavior of structures and invariably implies either a computational method for dimensioning parts or a design method for developing structural form. Such methods help translate theory into practice, but they do not suffice in themselves: the process of translation also requires a facilitator. Builders' "pocket books" often served as such facilitators. An interesting example was Carnegie's "Pocket Companion".

The Carnegie Pocket Companion[1] came into being in the late nineteenth century and it was the first practical American manual devoted to iron construction. It did two things extremely well: it segmented and packaged those parts of structural theory that related to iron and its use in construction in a form that practitioners could understand and apply simply, and it used that as a marketing tool to sell the firm's products. Originally issued in 1872, right at the very beginning of the transition from iron to steel in construction, the manual took the form of a company catalog and it ran for 65 years until 1937. The clever conflation of product catalog, simplified textbook, and construction recipes became "indispensable to users of structural iron" as the foreword to the 20th edition of 1919 proudly asserted.

The manual counted 24 distinct editions, but there is no record of how many variants the company issued, and it struck me that these might provide precisely datable information on how quickly new ideas and products penetrated practice. As a result I began tracing the editions, addenda, and variants, and have discovered a probable total of 39 to date, of which I have been able to examine 31. As every builder knows, "God lies in the details", whether the esteemed deity is an economic, a functional, or an aesthetical one, and I was rewarded by beginning to discover how the process of structural transformation from one building type to another functioned and how information was translated from theory and industry to practice especially with regard to the development of the steel-framed skyscraper in Chicago.[2] I am but at the beginning of this process, and there are many aspects that I have not yet recognized in their import, but I've already discovered a few ideas that are worth sharing. So this paper is a progress report, an exploration of the relationship between metallurgical and structural theory, the steel industry, and building practice. It is by no means an exhaustive study[3] or as yet, even a hypothesis of how the transformation or translation occurred.

Carnegie's manual was widespread in the United States and beyond. Between 1896 and 1919, the foreword to the latter edition claimed that Carnegie had distributed a total of exactly 238,686 copies: someone at Carnegie was evidently devoted to acribistic trivia![4] By the following edition of 1920 the number had grown to "about 254,000", and in 1921 to "about 275,000". All of this affirmed the manual's increasing popularity. According to an early article,[5] the issue of 1903 alone accounted for almost half of that.

For all its widespread popularity and the influence it exerted, Carnegie's Pocket Companion was not the first American catalog of rolled iron products. It did predate the flood of other industry publications in Pennsylvania by Bethlehem Iron, Cambria Iron, Jones & Laughlin, and Pencoyd to name only the most influential, but a series published by the Phoenix Iron Works began even earlier in 1869[6] and another by the New Jersey Steel and Iron Company in 1871.[7] In Europe, I am only aware of Charles Ferdinand Zorès's two editions of *Receuils des fers spéciaux* issued in 1856 and 1863,[8] but there were possibly others. While all of these were at first simple product lists that occasionally included test results to prove their reliability, the Carnegie Pocket Companion differed from the outset. It also differed from the nineteenth-century building encyclopedias, such as those of Peter Nicholson or Edward Cresy because it was directly tied to one material and to industrial production.

Like John Trautwine's more general *Pocket-Book* that also was not tied to any manufacturer or material and began its long run within the same year,[9] it was a true how-to manual of current practice and it offered much more than a simple catalog by translating up-to-date professional and theoretical knowledge into

practical terms.[10] A glance at the index of the Carnegie 1892 edition provides an example of the juxtaposition of theoretical, industrial, and practical knowledge that permitted the translation. It lists the short chapter on "Deflection and bending moments of beams under various systems of loading", followed by "Deflection coefficients for Carnegie shapes", and "Deflection limit to be allowed for plastering". In what has somewhat arbitrarily and exaggeratedly been termed Anglo-Saxon pragmatic fashion, it did not differentiate between theory and practice: it conflated them.

Trautwine commented on the problem of translation in the introduction to the tenth printing of the first edition in 1876:

> The writer does not include Rankine, Moseley and Weisbach, because, although their books are the productions of master-minds, and exhibit a profundity of knowledge beyond the reach of ordinary men, yet their language also is so profound that very few engineers can read them. The writer himself, having long forgotten the little higher mathematics he once knew, cannot. To him they are but little more than striking instances of how completely the most simple facts may be buried out of sight under heaps of mathematical rubbish.[11]

Trautwine was an eminent practitioner, so we must take his view seriously, and hc was not alone in his attitude. It was widespread, especially in the English-speaking world and even among the premier engineers of the time. We must remember that Telford preferred to suspend a full-length bridge chain and measure it, rather than calculate the curve by the mathematical means Gregory had provided: "With a practical man". his assistant Provis wrote in 1828, "an experiment is always more simple and satisfactory than theoretical deductions".[12]

And with regard to the failure of Navier's suspension bridge in Paris in 1827, Cumming writes:

> The engineer was called on for an explanation, when he said, "c'étoit seulement une petite distraction dans mes calculs". The ruins are removed, but the deed shall exist in the traces of our hand, that our countrymen may proudly shew that English modes of calculation, combined with practical skill are infinitely superior to being "initiées aux conoissances mathématiques les plus élevées". The reason is obvious; no man can make progress in the highest departments of mathematical learning who does not consume by far the greater part of his time in them; while with a certain degree of power in comparing quantities, and knowing the exact nature of the thing to be done, it is easy to make stability certain without having recourse to refined calculation.[13]

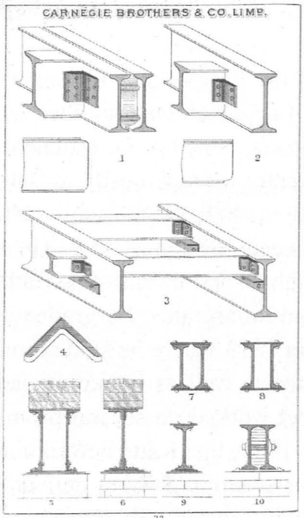
The celebrated remark, attributed I believe to Euler himself, that the parabola suffices as approximation of a hanging chain for a suspension bridge, although not polemically formulated like the preceding remarks, belongs in this mode of thought —and it is indeed true in this instance, because practical structural safety demands that building parts be overdimensioned, and this easily swallows such imprecision. Optimization, theoretical precision, safety, and the incertitude of imprecision and human error on a building site make uneasy bedfellows, and the practitioner is charged with the correct evaluation of each in any given situation.

Nevertheless, the latter third of the nineteenth century was the period in which intuitive and visual understanding, the pre-scientific understanding of structural behavior with its long and hallowed empirical tradition, no longer sufficed. It had to be supplemented by precise and quantifiable analytical knowledge in order to advance the field of iron construction. William Fairbairn was a leader in this movement, and his work, some of it together with Eaton Hodgkinson, contributed substantially to the nascent field of material science. The need for quantification gave rise to a new kind of translation problem, and, from the industrial standpoint, at least in the United States, there was also a need to instruct builders in proper procedure to forestall possible litigation against the manufacturer of a material should anything go wrong.

The Pocket Companion admirably fulfilled these goals and thereby promoted the spread of iron construction amongst North American practitioners. The second edition of 1876 did gather and paraphrase information from many sources including Rankine, Mahan, and Trautwine himself. The first edition (1872) probably used the two first mentioned as well, but I have not yet been able to consult a copy to prove it. The manual was assiduously updated from printing to printing, and it provides the historian with an excellent diary of the proliferation of structural knowledge and iron usage in the United States for the dynamic half-century, in the course of which modern steel construction developed.

The first edition was the brainchild of Walter Katté, an early American railway engineer and a man versed in many areas of construction practice. Born in England and trained there in a traditional engineering apprenticeship, he came to the United States at the age of twenty in 1850.

Katté worked for several railway companies, building track and bridges, and laying out a town in New York State. In 1865 he joined the new Keystone Bridge Works founded by Andrew Carnegie, the bridge builder Jacob Linville, and others.[14] Keystone became one of the mainstays of the Carnegie group and an outlet as well as advertisement for the iron and steel it produced. It also served as training ground for several of Chicago's most innovative skyscraper engineers. Katté's novel contribution was that he translated this business concept by combining a proprietary catalog with a general construction manual. He instructed



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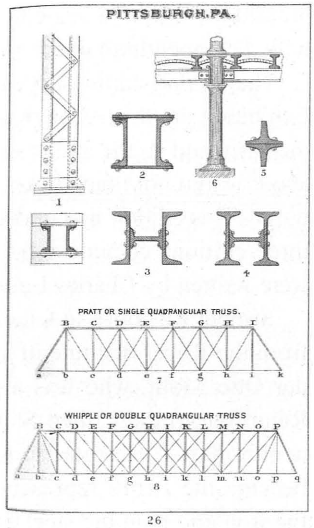

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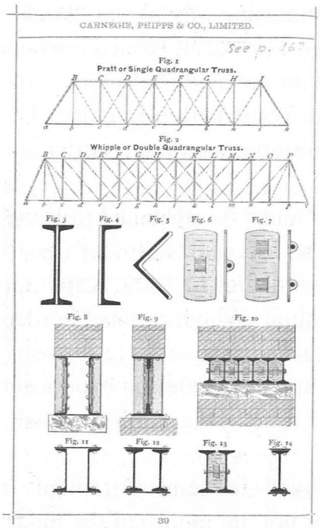

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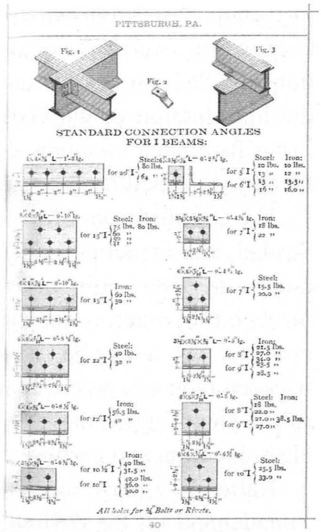

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Figures 1–4

When Strobel took over the editorship of the Pocket Companion in 1881, he began standardizing connection technology. At first, the connections were simple and not standardized. Gradually, however, he standardized and systematized his connection details. The transition from pragmatic detailing with different detail for each case to a standardized system approach was gradual, and it is clear which pages in the 1885 edition he retained from earlier versions and which were new. (Source: reissue of the 1884 Pocket Companion, dated January 1, 1885)

practitioners in the structural uses of iron while conveniently providing them with a compendium of the firm's products to use.

The second edition appeared in 1876, the year of the Philadelphia Centennial Exhibition at which American manufacturing came of age. Katté had by then left the firm, and so the author of this edition was an engineer named A. G. Haumann, about whom not much is known except that he apparently worked mostly in Pittsburgh. This edition was updated at least once, probably in 1880, but the subsequent three editions cemented the manual's influence between 1881 and 1890. These were written by Charles Louis Strobel, a man as farsighted and inventive as Katté.

Strobel was born in Cincinnati, the son of German immigrants. He graduated from the Royal Institute of Technology in Stuttgart in 1873 where he studied under Otto Mohr, who was a practicing railway builder as well as a theoretician. Returning to the United States at twenty-one, Strobel worked on several prominent bridges.[15] He came to Keystone in 1877, and in 1885, like Katté before him, became the firm's representative in Chicago. In this position Strobel delivered the iron and then the steel for the early skyscrapers and substantially contributed to technical solutions in construction. His great contributions to the "Pocket Companion" were the standardization of rolled steel shapes and riveted connections in 1881, an interest that his former teacher Mohr pursued in Prussia,[16] and the introduction of the German Z-bar to American construction practice. The Z-bar facilitated the many configurations of the built-up column and girder and thereby the modern skyscraper frame.

The establishment of the series and its popularity must be attributed to Katté, Strobel, and Frederick Henry Kindl (b. 1863), who had served as Strobel's assistant on the 1892 edition, and who took over the editorship in 1893. Katté and Strobel, (not much is known about Kindl), were cultural and educational border-crossers, which may be relevant to their transdisciplinary view of the relationship between industry, practice, and theory. They had gained their relevant experience as site engineers on the longest-span bridges of their time, and gave the manual a clear direction and purpose.

Katté may have consulted Trautwine who lived in semi-retirement in Philadelphia for the first edition. Whether he did or not, he modified the model and cleverly conceived the manual as a practical site "companion" and problem-solving tool. The success of the Pocket Companion was linked to its format and design. An engineer could use it at the office, but could also literally carry it in his pocket, and, like Phoenix before him, Katté bound a number of blank pages in at the end to serve as notepaper. The idea proved to be successful, and Carnegie retained the small, genuinely pocket-sized format for 35 years.[17]

Builders' pocketbooks have a tradition going back to sixteenth-century Italy, and one may perhaps stretch their genealogy all the way back to Vitruvius, be-

cause his treatise also provided useful information about construction methods, proportions, and other matters of practical interest.

What distinguished the Carnegie Pocket Companion from previous works, however, was that it contained exemplary and typical iron connections and details, up-to-date practically applicable analysis and construction hints, and it provided proprietary information as well as information on materials and structures produced by others, like the Roebling Company's iron floor systems, hollow-tiles for other forms of iron-supported floor systems, and most notably the cast- and wrought-iron proto-truss that James L. Jackson had designed in 1854 for his "fireproof" Harper Brothers Building in New York City (retained through the 1884 edition). This breadth of general knowledge and practical application established the manual's long-term influence.

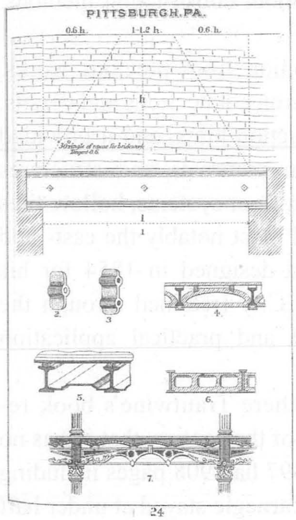
Carnegie adapted as the profession changed. Where Trautwine's book remained the same size but became so thick by the end of the century that it was no longer a practical pocket-book, (the 17th edition of 1897 had 908 pages including the advertisements and blank notepaper at the end), Carnegie stayed at under half that through careful editing, but chose to recognize the changing profession and publish a larger format from 1913.

The reason for this change was that design engineers had begun to remain more in the office by the early 20th century and they delegated site work to assistants. Thus, just before the World War, Carnegie issued its first medium-sized format, a size just a little too cumbersome to carry along in a pocket. At the same time the character of the manual gradually changed in the direction of a more traditional catalog, and the practical details disappeared in favor of product types. The new, mid-sized format complemented other contemporary mid-size Carnegie publications, such as the "Shape Book" which had begun publication in 1903, a new manual called "Standard Specifications", which first appeared in 1911, and "Beam Sections" from 1927. From the edition of 1913 on, the user was referred to the "Shape Book" for a complete listing of products, and in 1923, the specifications of the American Society for Testing Materials were moved to the "Standard Specifications". In this way, the Pocket Companion stayed a manageable size in contrast to Trautwine.

This mid-sized format lasted through the abridged 23rd edition of 1930. It was followed by a larger format in the second abridged edition of 1931 that was clearly intended only as a desk reference work, and it remained that size until the series ended in 1937.

I have identified three specific instances in which the Pocket Companion translated issues of theoretical import and industrial interest into practical terms, all of them under Strobel's or Kindl's authorship.

The 1884 edition exists in two versions. The earlier contains blank pages numbered 29–31 that were reserved for "Additional Shapes". The later has



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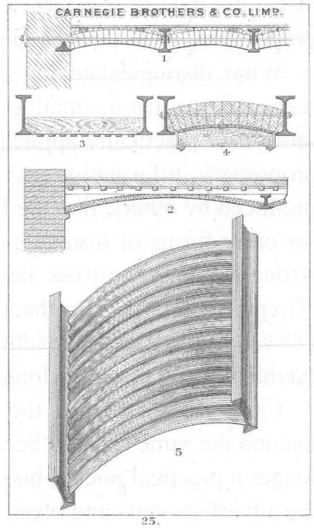

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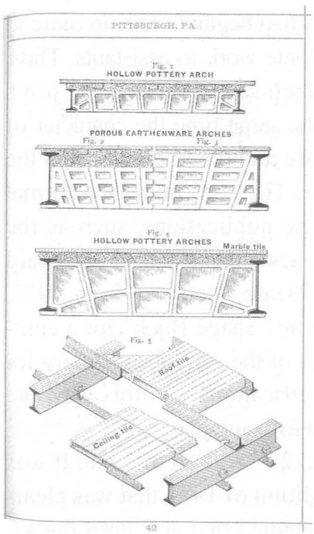

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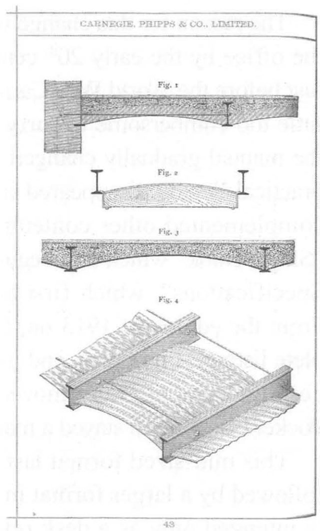

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Figures 5–8
Proprietary fireproof floor systems by other manufacturers helped show how Carnegie products could be used, and simple rules explained theoretical concepts to the non-academic practitioner. Like his standardization of details, these pages also became more sophisticated by 1890. (Source: reissue of the 1884 Pocket Companion, dated January 1, 1885, figures 5 and 6, and the 1890 edition, figures 7 and 8)

Carnegie's first "New Pattern Beams. Made in Steel only" on page 29. The altered page 31 that follows this bears the date Jan. 1 1885, so this variant is datable. Its importance lies in the fact that it documents that Carnegie began rolling steel I-beams in the course of 1884. The two I-shapes shown on page 29 were the first such members to be rolled in steel in the United States and perhaps anywhere. They must be those that Strobel recommended to William LeBaron Jenney for use in the upper floors of the Home Insurance Building in 1885 as the building was nearing completion.[18]

Eiffel, with whom Jenney studied at the École Centrale in Paris, was known to be wary of the new material. He never used it, not even for his Tower, apparently because he considered it too brittle. Jenney probably had the same reservations, but they were overcome by Strobel, who was not only a bridge builder who was familiar, at least to some extent with the new material, but editor of the Pocket Companion and Carnegie's representative in Chicago too. The use of steel in tall building frames dates from this period and has been taken by historians to mark the beginning era of the skyscraper. Although the few beams Jenney used in the upper stories of his building did not result in a steel frame, but were bolted to the existing cast-iron columns, their use does mark the first appearance of structural steel members in a tall building, and the myth of the "first skyscraper" stems from this fact.

The edition of 1890, was an important one and the last that Strobel edited alone. It marked a fresh departure in content and a closer connection to the skyscraper development in Chicago where buildings were growing taller by the year. It was only to be in 1894 that the city council enacted the first height limitations, and the rush to obtain building permits before the height limitation went into effect pushed the development even further. Carnegie could not roll column cross-sections that were large enough to bear eighteen stories, and engineers fell back on the bridge-building tradition of using built-up sections for especially large members. Strobel, who had spent the previous ten years streamlining Carnegie's production and standardizing the number of members in the firm's palette, invented a series of "Z-bar columns" in iron in 1888 and featured them in the 1890 edition.[19] A remark on page 133 identifies them as Strobel's design.

The Z-bar itself was not his invention, however; as far as we know, it had been first rolled in wrought iron by the Burbach Mill in Germany in 1862. Strobel may have seen its potential for building up larger steel sections than the mills could roll, and presumably imported the idea to the United States. He may even have first used it on Shaler Smith's High-Level Bridge as early as 1877. But whether or not it was he who introduced it, it first appears in Carnegie's catalog under his editorship. Strobel's position as Carnegie's Chicago representative allowed him to introduce the Z-bar there for built-up column sections. He reflected

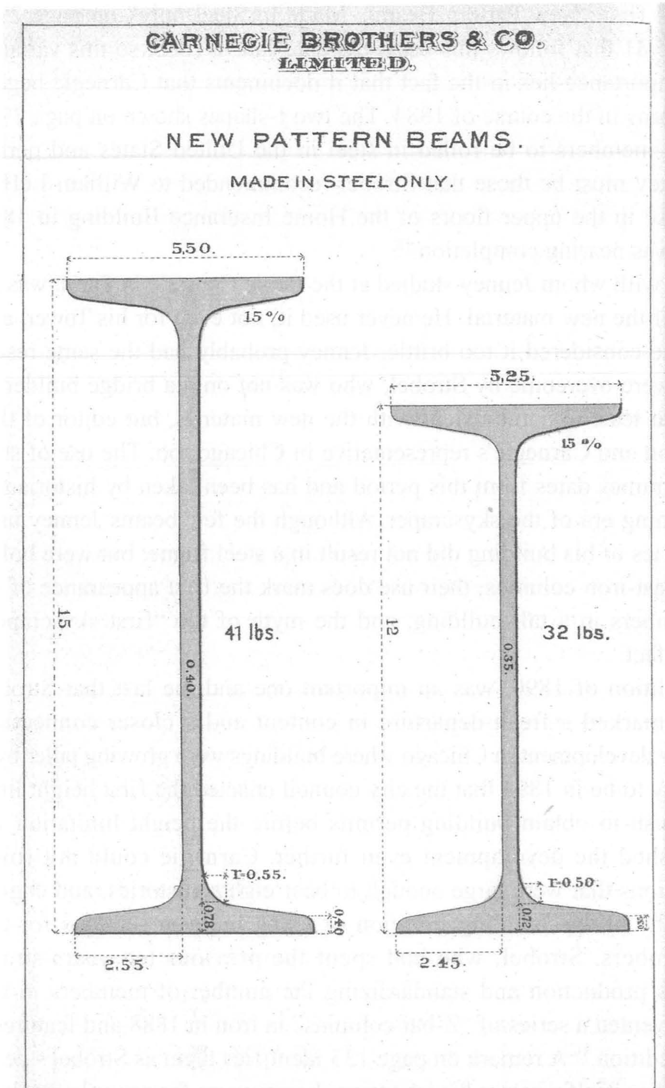


Figure 9
Early in 1885, Strobel added a page to the 1884 edition which showed Carnegie's new steel beams, the first I-sections to be rolled in steel in America and perhaps anywhere. These were the beams he offered to Jenney for use in the upper stories of his ten-story Home Insurance Building in Chicago, the first use of steel structural members in a tall building. (Source: reissue of the 1884 Pocket Companion, dated January 1, 1885)

that innovation in the 1890 edition, spreading the knowledge and the use of Carnegie's Z-bar throughout the United States wherever tall buildings were being built.

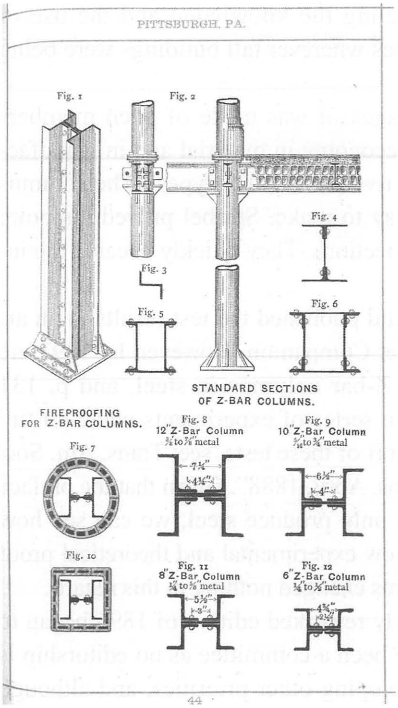
Strobel lists the Z-bar column's advantages: it was made of open members that permitted painting (and maintenance), economy in material and in manufacture (because z-bar columns required fewer rivets than other types). The columns resisted buckling, and connections were easy to make. Strobel proved his point by providing two pages of standardized connections. They quickly became the industry standard for tall buildings.

Strobel designed them in wrought iron and published the test results in an article in 1888 that he cited in the 1890 Pocket Companion. However, by 1893, no tests had yet been repeated on full-sized Z-bar columns in steel, and p. 131 states: "the above deductions are based on a series of experiments made on full sized *iron* Z-bar columns. For a detailed report of these tests, see: Trans. Am. Soc. C. E., paper by C. L. Strobel on Z-bar columns, April, 1888". Given that the preface also states that Carnegie would henceforth only produce steel, we can see how rapid the development was proceeding and how experimental and theoretical proof was playing catch-up. The 1894 and 95 reprints changed nothing in this regard.

In 1894 Strobel left the firm, and the fully reworked edition of 1896 began to lose its clear purpose under what must have been a committee as no editorship is attributed from then on. The firm was developing other priorities, and although the preface states: "Our product is exclusively steel", the comment about the lack of tests remains until 1900 when the problem had obviously been resolved.

The most interesting editions as regards the interchange between theory, industry, and practice were those between 1885 and 1896, the years that corresponded to the formative period in skyscraper construction. The complexities of the steel frame and its stiffness were among the most intractable issues. One of the problems was how to calculate secondary stresses in the connections, a problem on which Mohr was working at the time in Germany. The known brittleness of steel coupled with unknown stresses must have discouraged experimentation. In a way, iron builders' hesitation resembles the early English railway builders' untested fears that the friction between iron wheels and iron rails was too low to allow inclined tracks.

Strobel read German and knew Mohr, and he may have reassured the builders in Chicago that the problem was not as critical as they feared. He could point to several successful steel bridges, smaller ones that had followed the first at the Paris Exhibition of 1867, and a few large ones too —Fowler and Baker's great Firth of Forth Bridge was nearing completion.[20] The year 1890 that marked its inauguration also saw the first riveted skyscraper frame for the Rand McNally Building in Chicago.[21] The engineer was Corydon Purdy.



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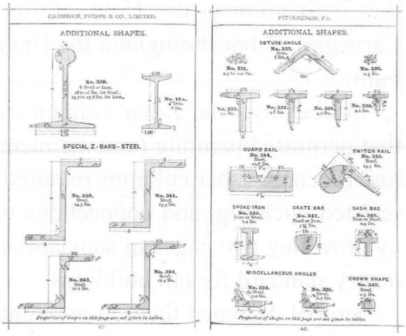

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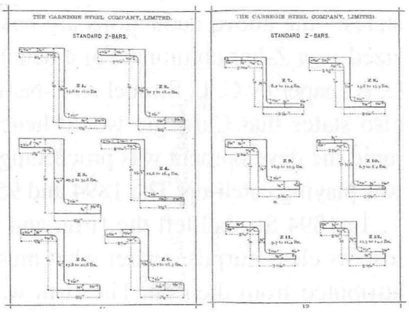


12

Figures 10–12
The German Z-bar appeared in Carnegie's catalog from 1890 on and supported Strobel's attempts at standardization by enabling built-up sections. By 1896, both the number of different Z-bars and the column types had proliferated mirroring the spread of the sky-scraper. (Source: 1890, figures 10 and 11, and 1896, figure 12, editions of the Pocket Companion)

Purdy gained his first professional experience as city engineer in Eau Claire, WI while working toward his Masters degree from the University of Wisconsin. In 1887, he moved to Pittsburgh as material inspector and designer for Keystone. Then, in 1889, he followed Katté and Strobel to Chicago, but opted for private practice, opening a bridge consulting office, quickly adding commercial buildings in partnership with Lightner Henderson. After designing the first steel frame for the architects Burnham and Root's Rand McNally Building in 1890, he attempted a complicated and therefore unrepeated diagonal stiffening system for Holabird & Roche's Venetian Building and then developed the successful portal-frame for

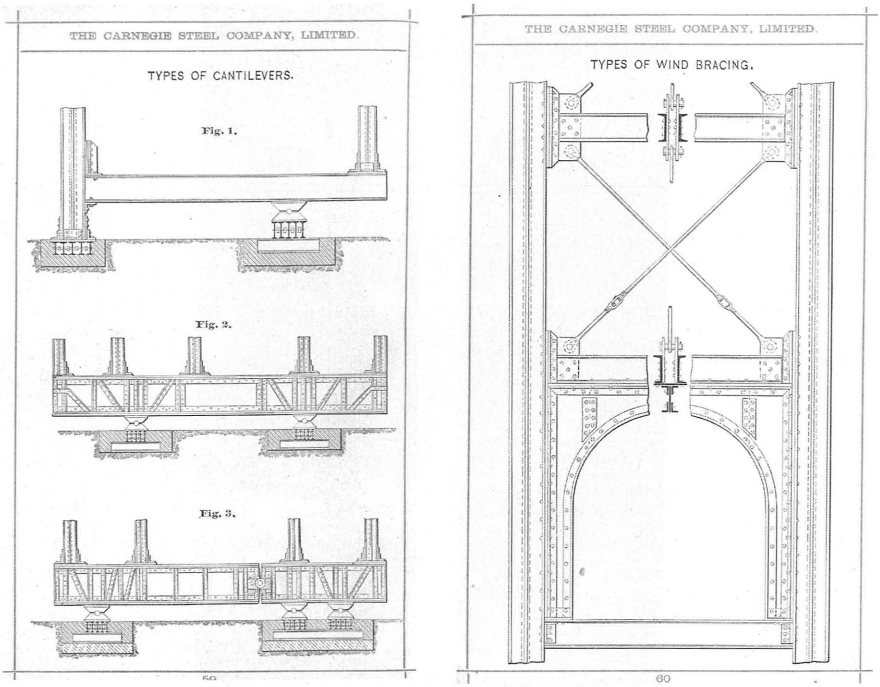


Figure 13
On page 59 of the 1896 edition, Purdy's cantilevered column that he had designed for the Old Colony Building in 1894 (at the top of the page), made it easier to build a tall building right up to an existing party wall. On the following page 60, Kindl showed both Jenney's wind-bracing for the Manhattan Building of 1890, a solution he had probably adapted from bridge construction, and Purdy's portal frame of 1893 for the Monadnock Addition that was a more appropriate solution because it did not block structural bays and permitted a freer planning of office space. (Source: 1896 edition of the Pocket Companion)

their Monadnock Addition in 1893 and their Old Colony Building the following year. We may suspect that Strobel, Purdy's former colleague at Keystone and now the agent who provided the steel, may have contributed to the solution. Purdy's portal frame became the first truly stiff framing system that allowed a free-plan configuration, contributing substantially to the commercial success of the skyscraper.

In 1894, possibly in reaction to growing regulatory strictures in Chicago, Purdy moved his firm to New York, and two years later began working with the Chicago contractor George Fuller there. Offices followed in many U. S. and

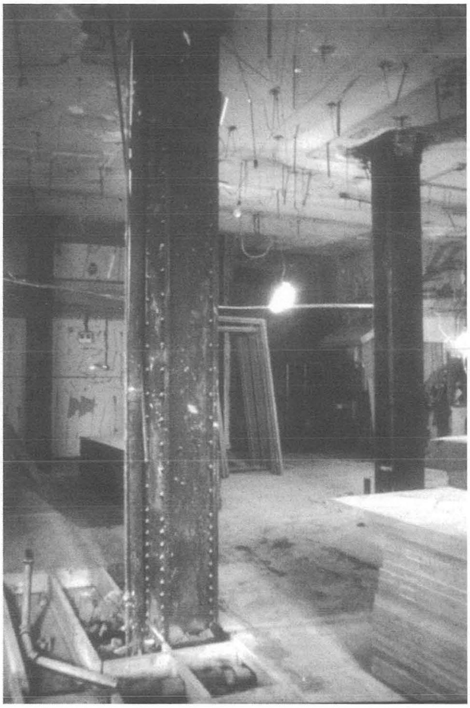


Figure 14
By the early 1890s, Strobel's Z-bar columns had become the industry standard. Here Chicago's Marquette Building of 1895. (Photo: T. F. Peters, 1983)

Canadian cities and in Havana. Purdy received the Telford Prize of the Institution of Civil Engineers in London and designed the influential exhibit of American construction technology for the Paris Exhibition of 1900. What interests us most is his successful portal frame system.

Like Jenney and others, Purdy must have been struggling with the issue of secondary stresses and stiffness. Diagonal bracing was limited by the need for functional flexibility. Blocked bays inhibited the free arrangement of floor plans, and after his expensive attempt in the Venetian Building to attach the diagonals' endpoints below the floor and above the soffit to gain space for doorways through the stiffened bays,[22] Purdy decided to use a portal frame. He did not invent it of course, but he hit upon the idea of importing it from the elevated Loop Railway running right beside the buildings he was constructing. Like Katté and Strobel's work, his importation constituted a transformation, not a translation

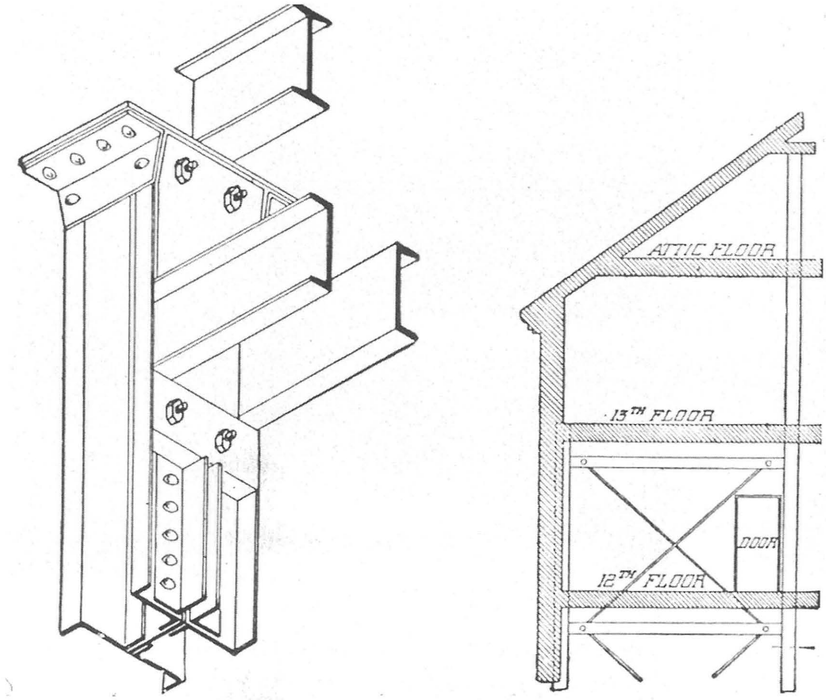


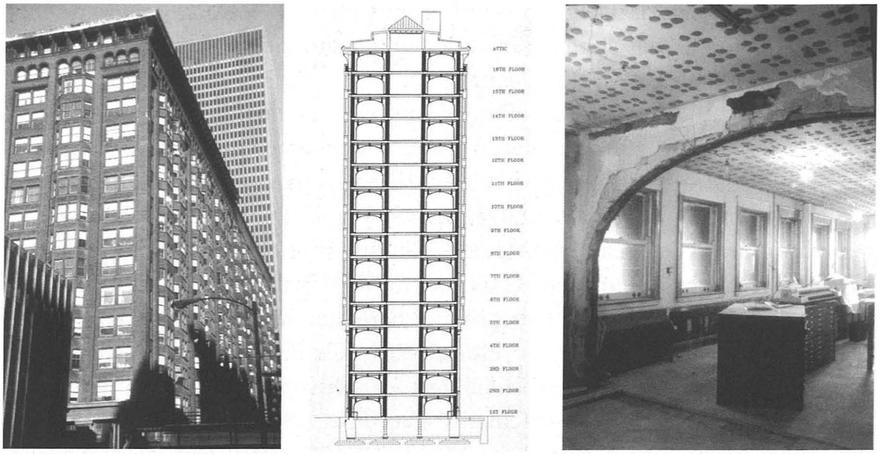
Figure 15
Purdy's stiffening system for the Venetian Building was short-lived. The system was complicated and expensive. (Source: J. K. Freitag: Architectural Engineering, 1906)

from theory or industry to practice but an application of the structural form to a new building type. It also constituted a translation of the form's behavior from that of a trestle that supports and stiffens a horizontal structure to a stacked mechanism for stabilizing a tall one.

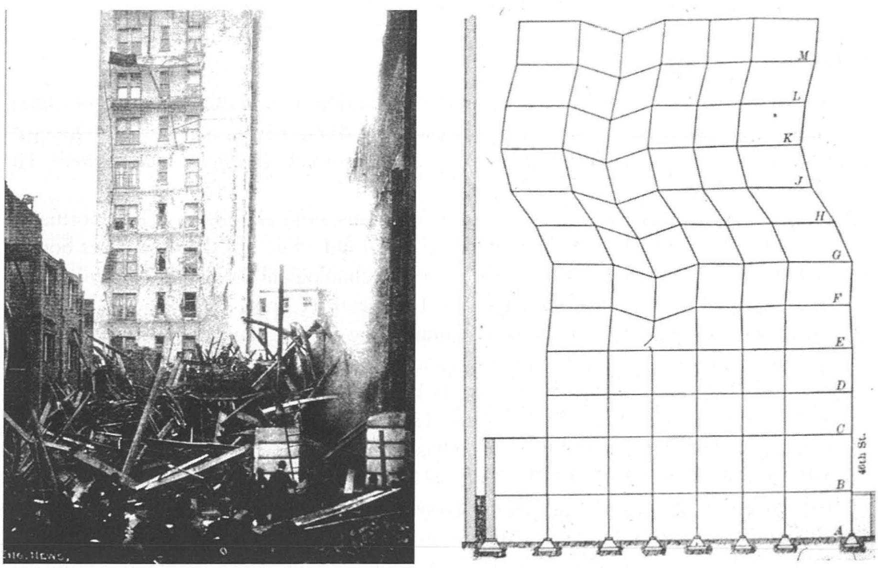
Using two lines of portals across a block gave sufficient stiffness and created enough redundancy in the connections to deal with secondary stresses too. This was about the time that Strobel left Carnegie for private practice, and Kindl featured the solution in 1896 on page 60 with a new explanatory article on p. 64, carrying forward the practice initiated by Katté of distributing knowledge of the best examples he could find as models to emulate. Wind-bracing in bridges was an old story. It had been an issue in the London Crystal Palace in 1850 too, but in Chicago Jenney had been the first to specifically note its importance in 1890 when he designed the Manhattan Building, the world's first 16-story, framed building. The diagonal bracing he used is also featured for the first time on p. 60 of Kindl's 1896 edition.

Figure 16
Purdy's portal frame stiffening was already being used in the El-railway in Chicago when he "reinvented" it for the Monadnock Addition. (Photo: T. F. Peters)

Purdy carried out a similar process of transformation for skyscraper foundations. In the Old Colony Building 1894, he cantilevered the columns adjacent to the southern party wall from the foundations of the neighboring columns. Economics dictated a maximum lot usage with every inch accounted for, so Chicago's tall buildings were built right up against existing, usually much lower party walls of unknown construction and stability. The concept of Purdy's cantilevering solution was known in iron bridge construction where Wilhelm Gerber had popularized it in Germany from the 1860s. Viollet-le-Duc had proposed it for buildings in his *Entretiens* in 1872,[23] and it had also long existed in wood, for example in medieval English hammer-beam roofs or hidden in the standard, light-wood housing framing system, called variously "balloon-framing" or "Chicago construction" that almost every American house was built of. So, once again, it was not an invention, but an adaptation, a transformation, and it was exemplary. It obviated the need for stabilizing existing party walls and the hazardous task of digging grillages for new column foundations under them, and Kindl presented Purdy's detail for emulation on page 59.



Figures 17–19
The Monadnock Addition of 1893 saw Purdy's first use of the portal frame stiffening system in a building (Photo: T. F. Peters)



Figures 20 and 21
The collapse of the Darlington Building in New York during construction in 1904 led to the abandonment of cast iron as a structural material. (Source: Engineering News, 1904)

Like the skyscraper itself, the Pocket Companion was a work in constant flux, reflecting the rapid development of structural understanding and the field of iron and steel construction. This is documented by the variations in and addenda to many of the editions with sometimes haphazard pagination, as well as by the swift transition from wrought- and cast-iron to steel production that accompanied the appearance of the modern steel skyscraper frame. Purdy had declared in 1890: "The use of cast-iron columns in 16–20-story buildings ought to be prohibited by law",[24] and Carnegie reacted. The foreword to the 1892 edition states: "The indication by different colors as [sic] to whether the section can be supplied in iron or steel." In 1893: "Our product hereafter will be exclusively steel". In 1896 "Our product is exclusively steel". It lies in the nature of industry to react more quickly to new knowledge than legislative bodies can, and it took the catastrophic collapse of the speculative Darlington Building in New York City that killed twenty-one workmen on March 2nd 1904, until legislators began to react and cast-iron was subsequently forbidden for structural purposes.[25]

There are more examples hidden in the pages of the "Pocket Companion", I am sure, but these few already suggest how the iron industry and its promulgators supported innovation in construction by transforming information from one area of structure to another and translating knowledge from theoreticians to practitioners and, even more importantly, translating between their modes of thinking.

Notes

1. Its full name, taken from the widespread 1903 edition, *was Pocket Companion containing Useful Information and Tables appertaining to the Use of Steel as manufactured by Carnegie Steel Company Pittsburg, Pa. For Engineers, Architects and Builders.* The name varied slightly over the years.
2. I have previously defined transformation and translation as follows: "Transformation remolds information within the boundaries of a field while translation crosses borders and moves it from one field to another." See: Technological thought is Design's operative method. In *Perspecta 31* (2000), 118–129, ills. MIT Press.
3. There is some obscurity about the early numbering of the Carnegie Pocket Companion, but the complete run, including every variant, reissue, and addendum appears to have been: 1872, 1876, a reissue with addenda, another with alterations around 1880, 1881, 1884, a reissue with alterations in 1885, 1890, 1892, 1893, unaltered reissue in 1894, another in 1895, 1896, a reissue with changes in 1897, 1898?, 1899?, 1900, 1903, 1907, 1913, 1915, 1916, 1917, 1919, 1920, an addendum in 1920, a supplement to 1920 (published in 1921), 1921, 1923, an extract of 1923, an addendum in 1926, another addendum in 1927, a further addendum in 1928, an abridged edition in 1930, a reprint of this in 1931, 1934, a reprint in 1936, a second reprint in 1937 (reissue = same date in imprint, reprint = new date). According to Thomas Misa, (T. Misa. 1995. *A Nation of Steel*, Johns Hopkins, 308, note 73), the Hagley Museum and Library appears to have a good selection. Aside from this the National Canal Museum in Easton, PA has a small

collection, the New York Public Library, Cornell University, and Yale University several early editions, and Lehigh University a somewhat more comprehensive run. If the dating according to R. Fleming's statement (History of Structural-Steel Handbooks. In *Engineering News* (March 8, 1917), 401–403), that 1892 was the 6^{th} and 1893 the 7^{th} is correct, (they provide no numbering), the three reissues of 1894, 1895, and 1897 appear to have been counted as editions, and there must have been two more in 1898 and 1899, because 1900 is marked as the 14^{th}. This also means that the reissues of 1876, one of them probably published around 1880, and the 1884 reissue published in 1885 were not counted as editions. The American Institute of Steel Construction publication (Herbert W. Ferris. 1953. *Historical Record Dimensions and Properties Rolled Shapes Steel and Wrought Iron Beams & Columns as rolled in U.S.A., Period 1873 to 1952 With Sources as Noted*. New York: AISC), appears to use an as yet undocumented 1889 imprint as a source (page 5), but this may be yet another uncounted reissue. Misa (1995, 170) refers to two dated 1901 and 1911, possibly reissues, that have not yet surfaced. Based on the forewords of the 16^{th} (1913) and the 17^{th} editions (1915), Fleming states the issue of 1903 to be a 15^{th} edition (thus discounting the existence of the 1907 as a separate edition), although the foreword in 1903 calls it "Edition of 1900. Revised 1903", discussing changes made since the edition of 1896, and therefore strongly suggesting that it was considered a variant reissue of the 14^{th} edition of 1900. By citing the edition of 1896 as the previous one, this discounts the variant of 1897 as a separate edition, and it indicates that there may not have been one in 1898 or 1899 after all, but this throws the whole numbering into disarray. It does look like there was a change in the numbering system around 1913, which creates confusion about the actual number. The ambiguity is increased by the variant reissues.

The materials directly consulted for this article were the 1876, its undated reissue and that of c. 1880, the 1884 and its variant of 1885, 1890, 1892, 1893, its reprint of 1895, 1896, its variant of 1897, 1900, 1903, 1913, 1915, 1916, 1917, 1919, 1920, the 1920 addendum, the supplement to 1920 (published in 1921), 1921, 1923, the extract of 1923, the 1926 addendum, the 1927 addendum, the 1928 addendum, the abridged editions of 1930 and 1931, 1934, and the reprints of 1936 and 1937. Moreover, reproductions of the title pages of the editions of 1872 (Fleming 1917) and 1881 (Frank A. Randall. 1949. *History of the Development of Building Construction in Chicago*, 33, figure 8. Urbana: University of Illinois). The antiquarian book dealer Julia Elton, Elton Engineering Books, London, cat. 6, 1991, item 174, listed the 1907 edition as a "16°", suggesting thereby that it was probably still the small, pocket size.

4. The Greek "acribia" means precision. It is used in German as "Akribie", meaning nit-picking, but in a value-free, non-pejorative sense. As I wrote in the compiler's note to the *Illustrated Bibliographical List of the Works of Louis Figuier…* (2002, 1. Bethlehem PA: Lehigh University Library), it deserves to enter our language if only to validate what I did there and have done again here!
5. Fleming 1917.
6. Phoenix belonged to the celebrated bridge builders Clark Reeves and Company, the developers of the Phoenix column. Their first catalog was titled *The Phoenix Iron Company, 410 Walnut Street, Philadelphia, Manufacturers of Wrought Iron Roof Trusses, either curved, straight, or hipped, Also, Wrought Iron Purlins & Jack Rafters, Arranged to suit Sheet Iron or Slate covering. Patent Wrought Iron Columns For Top Chord or Posts of Bridges or Piers, Depots, Factories, &c All Parts of Bridges, or Fire*

Proof Floors and Roofs, Made and fitted to suit designs of Engineers and Architects (1869), 24 p., only 7 of which contain tables and formulae, and 4 blank for notes. It was thus essentially a price list.

7. The iron founders Abram S. Hewett and Peter Cooper who rolled the first iron beams in America in 1854, published this catalog: Hewett, S.; P. Cooper. 1871. *Rolled Iron Beams Made by the New Jersey Steel and Iron Co*, 22 p.
8. Zorès is said to have been the manufacturer of the first rolled I-beam. His two catalogs were: 1856. *Receuil des fers spéciaux des expériences faites sur leur résistance et de leurs diverses applications dans les constructions avec notice*. Paris: Appert fils et Vasseur. 4 p, 36 (i. e. 38) pls.; and: 1863. *Receuil des fers spéciaux. II: Album; profils, assemblages, dispositions, armatures, suspensions & entretoisages des Fers Zorès, brevetés S. G. D. G., suivis de leurs diverses applications à la construction des sommiers, baux, poitrails [etc.]. Précédés d'une notice d'expériences comparatives sur la résistance des fers*. Paris: Dunod, 16 p.
9. John Cresson Trautwine (1810–83). The Civil Engineer's Pocket-Book, of Mensuration, Trigonometry, Surveying, Hydraulics, Hydrostatics, Instruments and their *Adjustments, Strength of Materials, Masonry, Principles of Wooden and Iron Roof and Bridge Trusses, Stone Bridges and Culverts, Trestles, Pillars, Suspension Bridges, Dams, Railroads, Turnouts, Turning-Platforms, Water Stations, Cost of Earthwork, Foundations, Retaining Walls, Etc., Etc., Etc. In addition to which the elucidation of certain important principles of construction is made in a more simple manner than heretofor*. 21 editions from 1872 (copyrighted 1871) to 1937. It is curious to note that both the Trautwine and Carnegie's Pocket Companion began and ceased publication at the same time.
10. There were other practical treatises, like that of Dennis Hart Mahan (1802–71) (D. H. Mahan 1837. *An Elementary Course of Civil Engineering, for the use of Cadets of the United States' Military Academy*. New York: Wiley), that built on his students' understanding of mechanics as taught at Harvard from around 1825 on by John Farrar (1779–1853), on the courses of Louis Charles Mary (1791–1870) on construction and public works at the École Centrale or of Joseph Matthieu Sganzin (1750–1837) (J. M. Sganzin. 1806. *Programmes ou Résumés des Leçons d'un Cours de Construction, Avec des Applications tirées principalement de l'Art de l'Ingénieur des Ponts et Chaussées, conformément au système d'enseignement adopté par le Conseil de Perfectionnement de l'an 1806*... Paris), at the École des Ponts et Chaussées, but those were all lecture courses rather than practical problem-solving manuals.
11. Edition of 1876, preface, x
12. Alexander Gibb. 1935. *The Story of Telford, The Rise of Civil Engineering*, 171. London: Maclehose, quoting William Alexander Provis. 1828. *An Historical and descriptive Account of the Suspension Bridge constructed over the Menai Strait*... London.
13. T. G. Cumming. 1928. *Description of the Iron Bridges of Suspension erected over the Strait of Menaï, The River Conway, and over the River Thames, also some account of the Different Bridges of Suspension in England and Scotland; with calculations on the strength of malleable iron, founded on experiments*, 51. 2nd. ed. London: J. Taylor.
14. Andrew Carnegie (1835–1919), the Scottish-born industrialist and philanthropist and Jacob Hays Linville (1825–1906) an eminent American railway and bridge builder. Walter Katté (1830–1917) represented the firm in Chicago from 1868, and in 1871 he

left Chicago for St. Louis to supervise the construction of James Buchanan Eads's (1820–1887) great bridge over the Mississippi. The following year, as soon as his Pocket Companion appeared, he left Carnegie and worked on further prominent projects both in St. Louis, (where he also served as city engineer for a year), and in New York City until his retirement in 1901.

15. Charles Louis Strobel (1852–1936) began his practical career with Louis Gustave Frédéric Bouscaren's (1840–1904) railway bridge over the Ohio River at Cincinnati in 1873. He then worked on the cantilevered, so-called "High Bridge" over the Kentucky River at Wilmore, KY designed under Bouscaren by Charles Shaler Smith (1836–86). Following its completion in 1877 he joined Keystone, becoming Carnegie's representative in Chicago in 1885. He left in 1894 to form his own firm, building many of the drawbridges over the Chicago River for the Scherzer Company (Albert H. Scherzer 1865–1916). Strobel retired in 1926.
16. Otto Christian Mohr (1835–1918) was by then in Dresden (from 1873), and his interests apparently resulted in the first Prussian standards for steel in 1884. It is not yet possible to ascertain whether Strobel influenced him in this.
17. The small-format editions (160–163 × 104–116 mm) were bound in two ways: a simple leather binding, which may have been thought of as an office version, and one with a practical internal pocket for loose notes and a fold-over flap with the company name embossed on it in gilt lettering. It is curious to note that Katté and Trautwine used the same format and the same flap-covered binding. Later manuals, like John Alexander Low Waddell's (1854–1938) *De Pontibus* also adopted it from 1898.
18. For a complete discussion of the role of William Lebaron Jenney's (1832–1907) Home Insurance Building (1885–1931) in the genesis of the skyscraper frame and the myth surrounding it, see Gerald R. Larson and R. M. Geraniotis's definitive article that explains how this came about and what is and is not true in the attribution of this building as the "first skyscraper": Larson, G. R.; R. M. Geranioti. 1987. Toward a better understanding of the evolution of the iron skeleton frame in Chicago. In *Journal of the Society of Architectural Historians 46*, pt. 1, 39–48.
19. The 1884 edition already has one "Z-iron", but it is not a true Z-bar with equal flanges appropriate for use as a connector. The 1890 has seven, four of which appear to be especially designed for use in Strobel's new Z-bar columns. Page 133 states specifically that Strobel had invented the Z-bar column. The reference to Strobel's article of 1888 on p. 133 indicates that these were introduced that year or the year before.
20. The Firth of Forth Bridge was designed by Benjamin Baker (1840–1907, knighted 1890) and John Fowler (1817–98, created baronet 1885) in 1881 after the original engineer Sir Thomas Bouch (1822–80) was removed in 1880 upon the collapse of the first Tay Bridge on December 25, 1879. The two engineers had previously designed an unbuilt, cantilevered Severn Bridge of steel in 1864 (1890. *Industries' Forth Bridge Special*, 4–5. London: Industries).
21. Architects were Daniel Hudson Burnham (1846–1912) and John Wellborn Root (1850–91), engineer was Corydon Tyler Purdy (1859–1944), and contractor was George A. Fuller (1851–1900). The building was demolished in 1911. No plans of the building survive.
22. This example was developed by Purdy for William Holabird (1854–1923) and Martin Roche's (1855–1927) Venetian Building 1892–1957. Crossing diagonals spanned from column to column across a bay, but instead of attaching to the beam-column connec-

tions, they ran from horizontal channels attached to the columns parallel to the beams, that lay about a foot below them. This meant that the diagonals disappeared into the floor and ceiling about thirty inches from the columns, providing just enough space to place a door. The channels were hidden above the suspended soffits. In this way the braced bays were not completely blocked and a certain amount of flexibility in planning was guaranteed. Needless to say this solution was too complicated and expensive to last.

23. Eugène Emmanuel Viollet-le-Duc (1814–79). 1863/72. *Entretiens sur l'architecture*, 3 vols. Paris: Morel. Especially vol. 2 (1872) *Dixhuitième Entretien*, illustration on page 328, and plate 36 (1863).
24. Frank Alfred Randall (1883–1950). *History of the Development of Building Construction in Chicago*, 16. 2nd. ed. Revised and expanded by John D. Randall. 1999. Urbana: University of Illinois.
25. *Engineering News* 50 & 51, March 10 & 24, April 14 & 15, May 12, 1904.

Gustave Magnel (1889–1955), a scientific biography

Patricia Radelet-de Grave

Gustave Magnel's vigorous, resolute and endearing personality, as well as his humour and pedagogical skills, appear in his numerous writings and papers. Born in Essen near Anvers on September 15th 1889 he died in Ghent on July 5th 1955. In his day he left his mark in several fields related to civil engineering. He was interested in every facet of this demanding profession and his life therefore

Figure 1
Gustave Paul Robert Magnel beneath a picture of one of his major works: the Sclayn bridge. (SECO Archives)[1]

shows a thorough image of the structural engineer from the first half of the 20^{th} century; a time when new techniques appear, partly due to him.

Beginnings

When Gustave Magnel gets his degree of Civil Engineer at the *Rijksuniversiteit* in Ghent in 1912, he has already written 7 papers on very different topics, among which the metal wearing due to abrasive materials, the metal welding, the ultra-violet rays, and some theoretical problems of mathematics. These early papers testify of the wide range of his interests. During the First World War he joins the D. G. Somerville Enterprises in London. He rapidly becomes a chief engineer, but Prof. François Keelhoff soon calls him back to Ghent University in 1919. The nine writings from this period focus on the themes of elasticity and calculus of arches. We can notice in one of these papers, entitled *Méthode rapide de calcul des lignes d'influence d'arcs prismatiques surbaissés à deux encastrements* (Magnel 1917b), the concept that will recur from then on: the mathematics of the mathematician are not similar to the mathematics of the engineer. He wittily explains his idea in a note on the teaching reform in engineering schools (Magnel 1950e).

> A force may be a vector for the mathematician, but for the engineer it is something that pushes or pulls.[2]

In 1948 he explains his concept more thoroughly to the science class of the "Royal Academy of Belgium" of which he has been a member since 1946.

> Unfortunately, in these days, only theoretical hyperstatic systems created by mathematicians were available. But mathematicians rarely worry about the time required to solve a numerical problem. Treatises from this period said that to solve a n times hyperstatic system, it was sufficient to write down n times that the derivative of the deformation strain with respect to each of the hyperstatic unknown quantity is equal to zero, and solve the n equations of n unknown quantities such obtained. They sometimes added that this calculus could easily be done with matrices. Let's imagine the poor engineers in front of a problem with, let say, 20 unknown quantities; it would have took them months only to solve the equations, and months only to prior write them down (Magnel 1948 m).

Magnel would always have the constant desire to provide professional engineers with simple mathematical tools that could ease computations. Nonetheless

he was perfectly aware of the developments of theoretical mathematics that he used to teach during his classes. In 1923 he became secretary first and then president of the committee of the Belgian Association for Standardization (the later Belgian Institute for Standardization) in charge of establishing the rules for the calculus and the building of reinforced concrete works. His first report already points out the question of the concrete shrinkage that generates tensions in the reinforcement: a problem which would grow more complex with the emergence of prestressing technology.

The academic carrier and the laboratory for reinforced concrete: teaching, research and experiments

Back to Ghent University, Magnel becomes head of the laboratory of material strength and assistant in the preparatory classes for Engineering Schools. He thus links, from the very first years of his carrier, experimental research with teaching. It is important to notice that all the activities of Gustave Magnel have always been strongly tied together and linked to the professional practice of the building engineer. In 1922 he offers a free course on handy calculus of reinforced concrete. This course did not exist in Belgium before and soon enough, the following year, he publishes the volume I of his famous book *The practice of reinforced concrete calculus* (Magnel 1923d). This volume, whose first edition is 160 pages long, is dedicated to the *practice of organic calculus*, the *Belgian requirements for reinforced concrete works*, (this part being the consequence of his work at the Belgian Association for Standardization), the *requests for cement providers*, the *theory of reinforced concrete*, and last of all, the *scientific dosage of concrete*. This volume will be re-published four times and will grow from 160 to 515 pages without the topics being changed. In the last edition of 1949, the various parts of the volume are always the same. Nevertheless, Magnel progressively includes in his volume the new experiments and knowledge enlargements in the different topics, whereas the new fields are progressively approached in the three other volumes of the whole *The practice of reinforced concrete calculus*. The second volume is dedicated to the analysis of *continuous beams*, *hyperstatic plates*, and the *composed warped bending*. The third one is about the problem of *arches calculus*, a problem that we also find in numerous papers, and finally the fourth volume is about *prestressed concrete*. This fourth volume comes out some time after the previous ones, and we will discuss it later on.

These volumes, some of which are completed by volumes of illustrations, are meant for a student readership and they gather all his personal expertise. In his teaching reform (Magnel 1950e), Magnel states that:

> Teachers ought to either publish their courses, either adopt the book of another author. Students should be forbidden to take notes; when you take notes you cannot think.[3]

Magnel's courses, however, largely illustrated with examples from real practice, show that an engineer does not become a professional only by reading books. In 1923 Magnel decides to put up a laboratory of research and training on reinforced concrete, and two years later he sets it in the ground floor of the Hotel Flandria Palace[4] near to the *Sint Pieter* station in Ghent.

In 1937 the laboratory moved to the *Technicum Complex* of *Sint Pieternieuwstraat*, figure 3. With Prof. Cloquet, Magnel personally and actively took part in the design of this technical complex, in the same way as he would later participate, in 1933, in the construction of the *Boekentoren* of Ghent University with the architect Henry van de Velde.

This laboratory, which later on moved to Zwijnarde, still exists and is now named after its founder. Magnel assigned it other tasks in addition to teaching. In

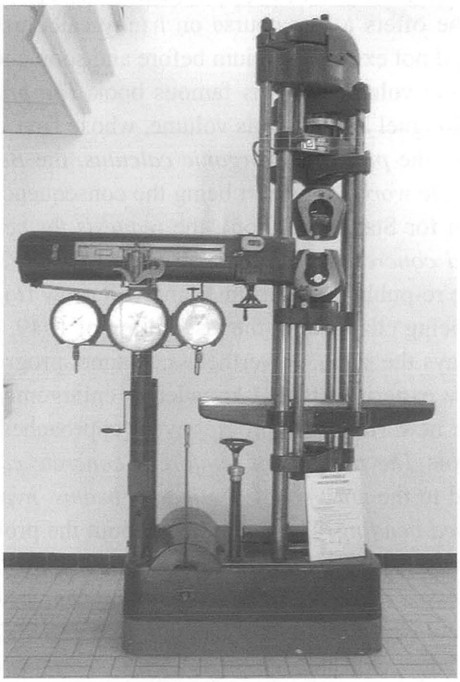


Figure 2
One of the two first testing machines[5] of Magnel's laboratory. Losenhausen (1926) universal testing machine, of 30 tons. (Photo P. Radelet)

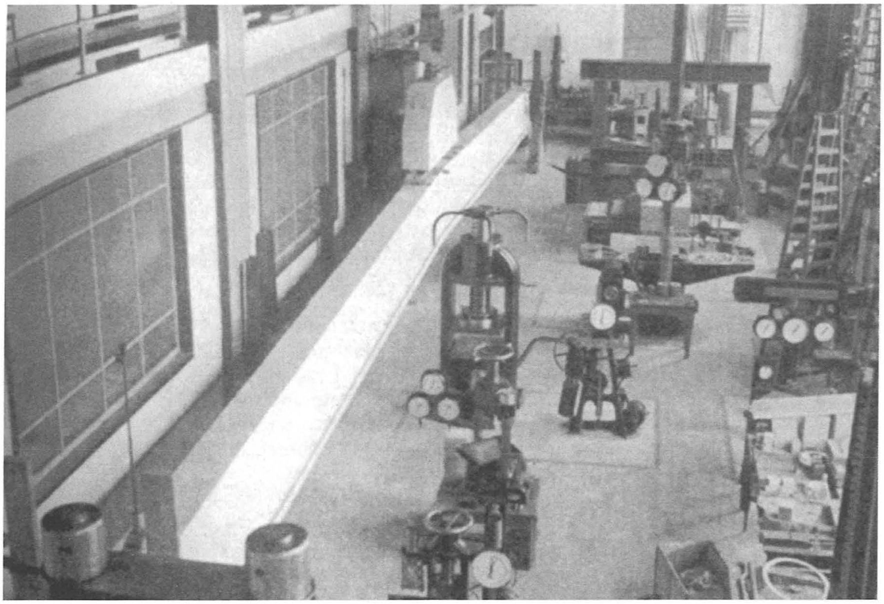


Figure 3
Interior of the laboratory of Sint Pieternieuwstraat. (Magnel 1953f)

fact, the laboratory was able to do some tests for third persons, at a certain fee; figures 6 and 7.

1° testing the received materials: cements, bricks, tiles, stones, concrete and concrete products.
2° tests aiming at determining the best qualities of some materials in order to produce concrete on big building sites
3° tests aiming at estimating the value —or not— of some new products such as special steels, waterproofers, etc (Magnel 1939j).

Later on we will give some more precise examples of these tests, replacing them in their original context. In addition, the laboratory is in charge of the control of all Belgian slag cement plants, Magnel being the president of their professional union.

In the meantime Magnel's academic carrier proceeds regularly. He becomes teacher in 1927 and then associate professor in 1932. Two years later he publishes a book on *the practical calculus of Vierendeel beams*, figure 8. Here again we

Figures 4 and 5
Two pictures of the second of the two first testing machines[6] of Magnel's laboratory. Losenhausen press (1926) of 300 tons. (Photo P. Radelet)

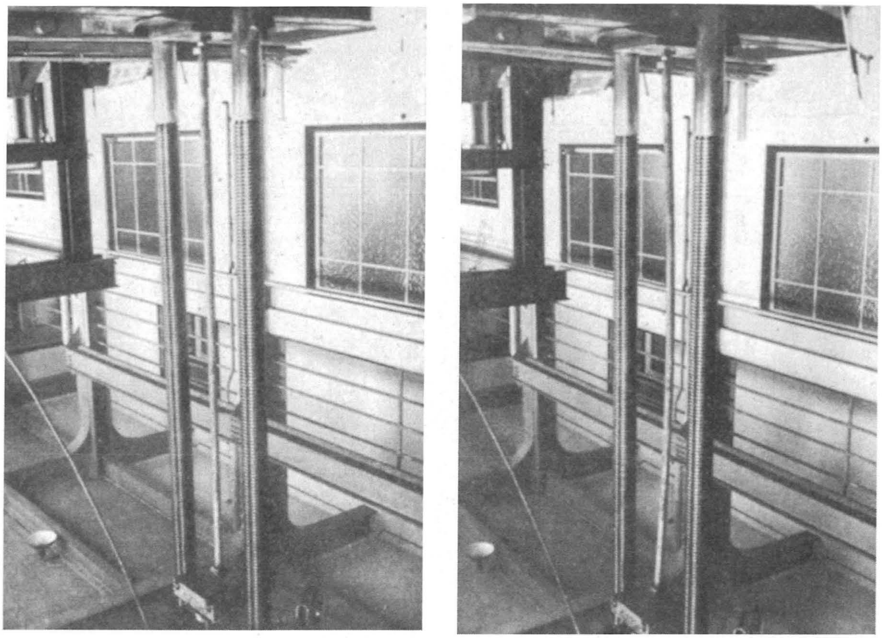
recognize his desire to make engineers' computing problems easy. This time he no longer deals with reinforced concrete but with special metallic beams that do not have diagonals, according to Vierendeel's patent (Radelet 2002 and Radelet 2003). They thus are non trussed beams. Unlike his previous books, this one will not be re-published. Metallic beams are substituted, at least in Magnel's concerns, by prestressed concrete beams.

In 1935 he publishes the first volume of another series of four, which will be re-published several times, similarly to the volumes on the practice of calculus of reinforced concrete. It is the series of the *Course on Constructions Stability* (Magnel 1938).

Magnel becomes full professor in 1937.

The *SECO*: building sites security and insurance

On December 21st 1934, Magnel founds, together with Eugene François, Professor at the *Université Libre de Bruxelles*, the *SECURITAS* bureau, father of the present *SECO* bureau, following the example of the SECURITAS office



Figures 6 and 7
Buckling test in the laboratory of *Sint Pieternieuwstraat*. (Magnel 1953f)

which had been founded in Paris in 1927. The society's first charter explains its goals:

> The goal of the bureau is to operate technical control on the projects, the computings and the execution of the works in building industry and civil engineering, in exchange

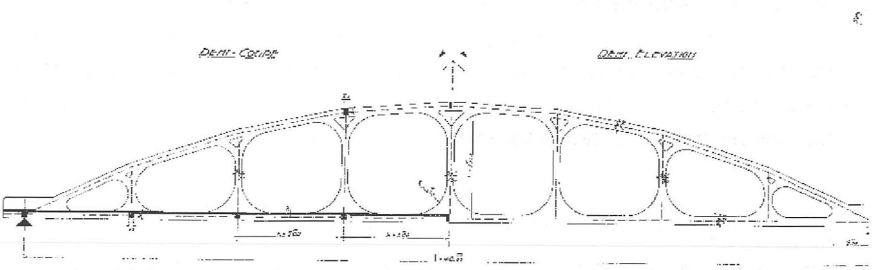


Figure 8
Vierendeel beam in Magnel's book on *the practical calculus of Vierendeel beams* (1934)

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A tous ceux qui désirent construire.

" L'architecte exerce peut-être la plus dangereuse des professions. De son côté, l'entrepreneur expose quotidiennement son patrimoine et l'avenir de sa famille. Le maître de l'ouvrage, enfin, risque de traîner derrière lui un boulet jusqu'à la fin de ses jours.

Maître HILBERT, avocat à la cour d'appel de Bruxelles
" Responsabilité du bâtisseur ".

N'oubliez pas le régime d'exceptionnelle rigueur et la responsabilité écrasante et de longue durée qu'entraîne l'application des art. 1382, 1792 et 2270 du Code Civil.

ART. 1382. Tout fait quelconque de l'homme, qui cause à autrui un dommage, oblige celui par la faute duquel il est arrivé, à le réparer.

ART. 1792. Si l'édifice construit à prix fait périt en tout ou en partie par le vice de la construction, même par le vice du sol, les architectes et entrepreneurs en sont responsables pendant dix ans.

ART. 2270. Après dix ans, l'architecte et les entrepreneurs sont déchargés de la garantie des gros ouvrages qu'ils ont faits ou dirigés.

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QUELQUES RÉFÉRENCES

Fondations et pylônes de la ligne expérimentale à Zeebrugge de la Commission spéciale de l'étude de l'action du vent sur les constructions, de l'Association Belge de Standardisation.
Entrepreneurs : Sté Engeira et Sté Métallurgique d'Enghien St-Eloi.

Magasins et Immeuble de rapport, Boulevard d'Ypres, à Bruxelles.
Architecte : M. DE BOELPAEPE.
Ingénieur-conseil : M. ROSSBACH.
Entrepreneur-Général : Société Anonyme LOUIS DE WAELE.
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Entrepreneur-Général : A. MONNOYER et E. FRICERO S. E. N. C.

Trois Villas, à Ath.
Architecte : M. A. BROQUET.
Ingénieur-conseil : M. HAUTIER.
Entrepreneur-Général : M. L. BROQUET.

Maison de l'I. N. R., Place Sainte-Croix, à Ixelles.
Architecte : M. J. DIONGRE.
Ingénieur-conseil : M. CHAPEAUX.
Architecte-général : M. R. GILLION.
Fondations : PIEUX FRANKI.

Magasins et Immeuble de rapport, Rue Washington, à Ixelles.
Architecte : M. DELALIEUX.
Ingénieur-conseil : BUREAU HENNEBIQUE.
Entrepreneur-Général : MM. HAMBRESIN FRERES.

Figure 9
First presentation of *SECURITAS* office. (SECO Archives)

of a fee. Control aims at reducing the risk of accidents and at making possible the widest expansion of insurance policies at reduced costs.

. . .

The control service is optional, even for the members of the bureau. No one is required to submit; the bureau does not provide design and drawing services nor does it conduct construction operations. It only gives advice and suggests cautiousness on the base of scientific knowledge and expertise. Therefore it does not take on pecuniary responsibilities (SECO Archives).

The presentation booklet published by SECURITAS in December 1935, figure 9, reminds that Belgian law is extremely strict towards architects and contractors. The latter therefore deserve to be provided with good insurances.

Art. 1382: Any human act which causes damages to anyone compels the responsible of the act to repair it.
Art. 1792: If a construction built with forfeit fees gets ruined in total or in part due to poor construction techniques, or even to poor ground, architects and contractors are responsible for ten years.

Art. 2270: After ten years, architects and contractors are relieved from the guarantee of the buildings that they achieved or directed.

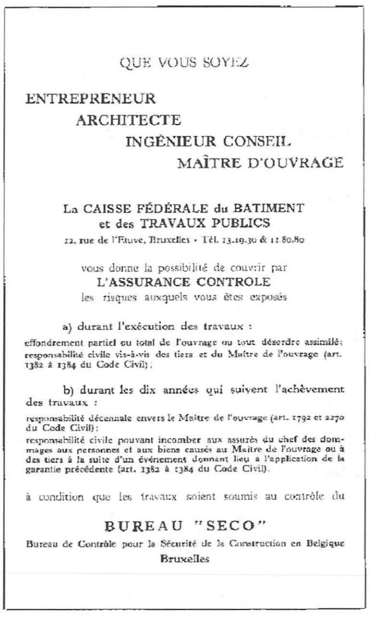
This booklet also releases information about the composition of the bureau.

The "Bureau Securitas" stems from a large association between architects, contractors, consultant engineers, joined by a technical committee formed by professors who teach stability and construction in all our Universities and at the Military School.

Magnel is a consultant of the Bureau and his laboratory does tests on structural components, especially when innovative procedures are involved, thus making possible the insurance of the building process.

The Cancer Institute of the Public Health System in Brussels, the Koekelberg basilica, and the North-South railroad junction are among the first works submitted to the control of the SECURITAS Bureau. The two latter works are among Magnel's achievements.

The full assembly of the *SECURITAS* Bureau gathers on May 12th 1938. 125 industrials, engineers, professors, architects and contractors are present, for a to-



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les risques auxquels vous êtes exposés

a) durant l'exécution des travaux :

effondrement partiel ou total de l'ouvrage ou tout désordre assimilé; responsabilité civile vis-à-vis des tiers et du Maître de l'ouvrage (art. 1382 à 1384 du Code Civil);

b) durant les dix années qui suivent l'achèvement des travaux :

responsabilité décennale envers le Maître de l'ouvrage (art. 1792 et 2270 du Code Civil);
responsabilité civile pouvant incomber aux assurés du chef des dommages aux personnes et aux biens causés au Maître de l'ouvrage ou à des tiers à la suite d'un événement donnant lieu à l'application de la garantie précédente (art. 1382 à 1384 du Code Civil).

à condition que les travaux soient soumis au contrôle du

BUREAU "SECO"
Bureau de Contrôle pour la Sécurité de la Construction en Belgique
Bruxelles

Figure 10
Advertisement for the SECO bureau. (SECO Archives)

tal of 555 social parts among the 588 that had been distributed. They decide to change the name of the bureau. *SECURITAS* becomes *SECO*, control bureau for construction security in Belgium, figure 10.

During the war, while the construction business is struck by a severe recession, Magnel dedicates himself intensely to the SECO bureau. The bureau is challenged with the problems of building sites insurances in war days. In 1941 insurance companies ask for a 20 % raise of the insurance primes. A long negotiation begins between Magnel, SECO, insurances and re-insurances. Magnel tries to set up a handy classification of the different risks that may occur, in order to ease to fix the insurance fees for architects and contractors.

> The SECO bureau is working on the preparation of a code thanks to which a risk factor could be assigned to the different kinds of risk that insurance companies are confronted with (SECO Archives).

The classification will be finished in 1942. It includes six categories.

> II classification project of the risks according to their evaluation
>
> – *1st category*
> Houses
> Condominiums
> Private buildings for offices, apartments or theatres
> except for: those of more than 14 levels or higher than 80 m or having spans longer than 25 m
> a) less than 4 levels
> less than 25 m high
> spans shorter than 15 m
> b) more than 4 levels
> higher than 25 m
> spans longer than 15 m
> – *2nd Category*
> Public buildings except for: towers, churches and buildings higher than 80 m or having spans longer than 25 m
> a) less than 25 m high, with spans shorter than 15 m
> b) towers, churches and buildings between 25 m and 80 m high with spans shorter than 25 m
> – *3rd Category*
> Permanent industrial plants
> less than 60 m high (80 m for chimneys) with spans shorter than 25 m

a) everything, except for towers, reservoirs, silos, chimneys and special buildings
b) towers, reservoirs, silos, chimneys and special buildings

– *4th Category*

Big structures

except when pier walls or supporting walls or abutments' support is more than 10m deep, and when bridges have spans longer than 50 m

a) Any kind of bridge
b) 1° Piers or supporting walls bordering public roads that are not provided with machines for maintenance
   2° Piers in ports and similar works that are meant to support maintenance machines
c) sewage, aqueducts and similar works

– *5th Category*

Transformations

with exception in height and identical span to the corresponding transformed category

A; involving vitals of extant constructions

a) when the ten years guarantee amount is based on the sole cost of the transformation works
b) when the ten years guarantee amount is based on the cost of the transformation works in addition to the estimated value of the building or the parts of the building that are being transformed

B; not involving vitals of extant constructions

a) when the ten years guarantee amount is based on the sole cost of the transformation works
b) when the ten years guarantee amount is based on the cost of the transformation works in addition to the estimated value of the building or the parts of the building that are being transformed

– *6th Category*

Works that are not included in the first five categories, i. e.:

1st Category: more than 14 levels or higher than 80 m or spans longer than 25 m

2nd Category: higher than 80 m (100 m) and spans longer than 25 m

3rd Category: temporary industrial plants, constructions higher than 60 m (80 m for chimneys), with spans longer than 25 m

4th Category: piers, supporting walls, abutments etc, more than 10 m and spans more than 50 m

5th Category: same as the corresponding category that includes the transformed construction, and all special works, in particular soldered constructions, prestressed concrete, innovative designs, etc. (SECO Archives)

This classification is actually the first step towards the present agreement procedure for contractors. The present procedure differs from this previous one, since the agreement is nowadays given to the contractor himself, while here the risk is estimated for every single construction process. Furthermore, the present agreement has two clauses: a financial clause and a building-kind clause. Out of the financial clause, the contractor is allowed, according to his category, to build works whose cost is not superior to a fixed amount. The building-kind clause is similar, in essence, to Magnel's concept but because of the technical progress it was obviously largely completed during the following years.

The foundation of the SECO bureau by Gustave Magnel and Eugène François is indeed a great achievement whose originality must not be underestimated. A deep knowledge of the construction world and business was necessary to dare to start up such a project.

The prestressing technology

In 1950 Magnel wrote:

> Frequently enough, some people —not much aware of the last building techniques evolution— ask us if we are the inventors of prestressed concrete.
>
> We always answer that this new material does not have a precise father; the concept is more than 50 years old and we could easily mention several patents from the end of last century, related to prestressed concrete. (Magnel 1950f)

It is also during the Second World War, in 1941, that Magnel starts to study prestressed concrete; five years after the publication of the paper by Freyssinet *A revolution in concrete technology*: a paper in which the author introduces the *notion of prior stressing or pre-stressing*. Magnel actually recognizes the anteriority of the discovery by Freyssinet, with whom he had a very friendly relationship. He openly and honestly acknowledges Freyssinet's merit.

> It is nowadays commonplace to say that Mr Freyssinet is the inventor of prestressed concrete, nonetheless it is not exact. The concept of prestressing is as old as reinforced concrete itself, and there are several patents related to this topic even though Mr Freyssinet did not care. The success of this French scientist lies in the demonstration that it is impossible to realize a pre-stressing which won't fade in time without using steels of a very high elastic resistance, able to support tensions of about 80 or 100 kg/mm^2. We are not pointing out such facts in order to reduce Mr Freyssinet's awesome merits, but to restore historical truth. It is Mr Freyssinet who made prestressing possible in every day engineering art. He invented a technique and some equipment that are commonly used in France (Magnel 1949i).

In order to understand this merit and then appreciate Magnel's, we have to know better what prestressing is.

> Concrete is a material whose compressive strength is several times superior to its tensile strength (Magnel 1953d, 3).

The concept of reinforced concrete is thus to compensate this lousy tensile strength by having steels tensed. The disadvantage is that when steel is stressed, structures surely hold but concrete cracks. Let's listen to the teacher while he explains what to do:

> Imagine that we could put a concrete beam —eventually a not reinforced one— under a general compression equal in every direction up to, for instance, 150 kg/cm^2.
>
> If we then apply an exterior load to this beam, it will not crack until in each of its facets this load does not create a tension superior to the value of 150 kg/cm^2 plus the value of the concrete tensile strength.
>
> This is what prestressed concrete is all about. The general compression of 150 kg/cm^2 is the *pre-stressing* (Magnel 1953d, 7–8).

It is therefore necessary to compress the concrete strongly enough so that the tensions applied during its positioning on site will not decompress it enough to put it under traction. The following diagram illustrates this concept, figure 11.

Once again Magnel explains facts in a very entertaining way.

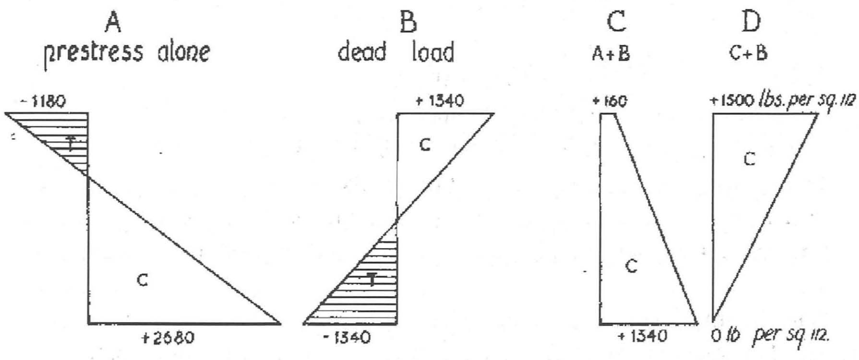


Figure 11
Diagram of the stresses inside concrete. A prestressing, B concrete weight, C sum of the weight and the prestressing, D sum of the weight, the prestressing, and an external load which is here equal to the weight. (Magnel décembre, 1945–janvier 1946)

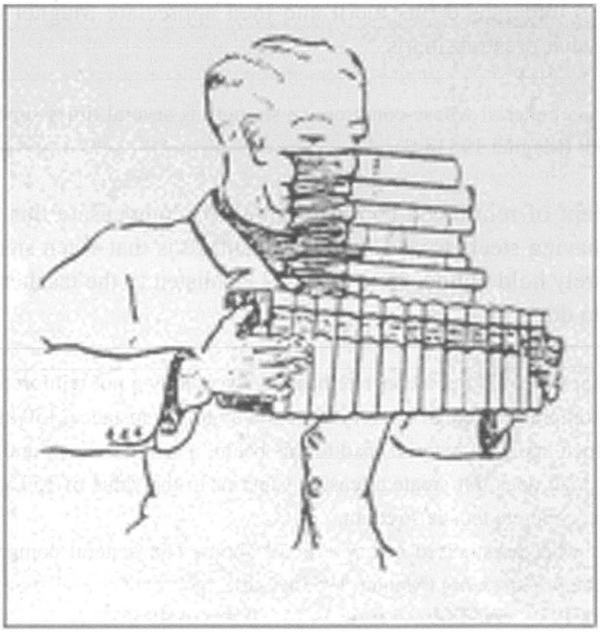


Figure 12
Magnel explaining prestressing technique

> All of you already practiced prestressing when you moved a bunch of books from one shelve to another. You had to press the books with your hands, but when you wanted to move too many books in once, it did not work; you were no longer able to apply a compression strong enough to balance the tensile force due to the books weight loading the beam formed by the books themselves. There is an ultimate solution for those who do not respect books: you can drill a hole through the books and insert a metallic tie in it with two fine nuts; by tying the nuts the books can be compressed and the "beam" can be put on two supports. The tie action is the "pre-stress", i. e. a stress which is applied prior the upcoming of the stresses of external loads (Magnel 1948m).

But the shrinkage of concrete while drying, and the creep, an irreversible plastic deformation either of the concrete or the steel, add some complexity to this simple idea, and progressively weaken the prestressing. This is the object of Freyssinet's researches. Let's read how he explains, in 1936, the concept of prestressing.

A good use of high resistance materials is thus due: for steel, to the maintenance of the concrete lengthening inside handy usual limits; for concrete, to maintaining the total tension stress below a limit much inferior to the tensile strength of concrete.

Theoretically, this double condition can be fulfilled by using steel, not to support the tensions and elongations that concrete cannot bear, but using it in such a way that it permanently stresses the concrete in the opposite direction as the loads stresses do; i. e. compression in the tensed parts, and reciprocally tension in the compressed parts.

This result may be attained by tensing the steel before casting concrete. In this case, reinforcement extremities are temporarily anchored and tensed by a known stress, thanks to jacks supported by some abutments. We then wait for the stiffening of the concrete, enough for the reinforcement extremities anchorage, and next the jacks are released. The reinforcement which tends to shrink applies to the concrete a compression equal to the tension. Another procedure consists in applying a tension to some steel ties by grabbing support on already hardened concrete.

In this way we create in the steel-concrete organism a double system of permanent identical stresses, but oriented in the opposite direction as the one that appears in usual reinforced concrete, due to the well-known bad consequences of shrinkage. The first result of creating a prestressing is thus the elimination of the harmful effects of shrinkage. For a long time, in many countries, practical means to achieve this goal have been searched for. But all these efforts failed and were abandoned, except for some applications to pipe fretting. (Freyssinet 1936)

The Magnel-Blaton patent

We now have good ingredients to produce prestressed concrete, but still the prestressing has to be done. Magnel and Freyssinet both invented equipments to compress concrete, thanks to steel wires strongly tensed. Freyssinet's is the earlier method but war brings Magnel to develop his own system.

Since it was impossible during the war to get Freyssinet's equipment, and we were willing to improve, if possible, the work of this great scientist, we helped to build some other kind of different device and created the sandwich wire (Magnel 1953d, 21–22).

The two procedures are thus similar. Let's study the Magnel-Blaton procedure first, and we will discuss the differences afterwards.

The Magnel-Blaton system

This procedure, based on the use of highly elastic resistant steel, differs from other systems by its own following characteristics.

1) Wires arrangement
   The wires of every cable are 4–5 mm distant from one another and arranged in layers of 4. Plain grids maintain the arrangement and the parallelism of the wires all along the cable's length.
   Thanks to this arrangement, positioning some special grids at the main deviation points is easy, and this will make possible the control and the reduction of friction.
   Furthermore, the covering of the prestressed cables will be easy and reliable.
   Laboratory experiments have shown that thanks to the perfect cover of each wire, the resulting adherence would be such that the beam could be loaded, after the release of the anchorages, up to its service load, without noticing any sliding of the wires.
2) These anchorages are metallic and their quality and precision are therefore easy to check
   They are shaped as a parallelepiped and made of high resistant steel, with two notches on each of the main faces.
   Two wires will be placed in each notch and fixed with a small steel wedge.
   Each anchorage (named sandwich) thus holds eight wires and by superimposing several pieces, the desired cable can be fixed.
   Partial prestressing can easily be done, since the wires are anchored by pairs, some of them may be put apart temporarily.
   A thick cable is easily manufactured by simply superimposing a great number of sandwich plates.
   A steel plate with a central rectangular hole comes between the sandwich plates and the concrete and helps to distribute on its whole surface the compression strength applied by the cable.
3) Tension by pairs:
   Thanks to the notch-wedge system, a jack seizes two wires and applies the desired tension and elongation.
   It is connected to an oil pump provided with a manometer for control. This very light device, since it only has to apply small loads, can be easily manipulated.
   Due to its light charges, it is possible to couple it with a dynamometric device which controls very precisely the tension applied to each pair of wires.
   The jack is equipped with a balancing system in order to balance the load applied to both wires and ensure equal tension in each.
4) Removing the metallic ducts
   Expensive metallic ducts around the cables, preventing the evaporation of the humidity surplus of the injection product, have been abandoned a long time ago.
   Rubber bars of a rectangular section are inserted in the lining before concrete casting.

They are removed one or two hours after casting; let's point out by the way that a 400–500 kg force is sufficient to pull them out since friction is not the problem but rubber elasticity.
Cables, manufactured in advance, beside the beam, are then introduced in the hole left by the rubber bars.
When concrete has acquired enough resistance anchorages are placed and tension is applied. Let's notice that these anchorages can be very easily positioned, between the wires layers which extend outside the beam.

5) Beams with exterior cables
Because of the rectangular section of the cables, it is possible to place them outside the beams, on each side of the web.
Horizontal in the central part of the beam, the cables are deviated towards the top of the extremities thanks to hooks that cross the beam's web. The hooks have special notches to receive them.
Furthermore it is possible to prepare the concrete in several segments, which are next aligned, joined together with cement mortar, and assembled by the prestressing.
The advantages of this procedure are:
the possibility to precast these segments at the plant, for a better quality control
the lack of congestion on building site
the easy positioning of the beams: the segments can in fact be put directly in their final position, the alignment being done and the tension being applied up there.

Advantages of the Blaton-Magnel System

1. Tension by pairs, making possible
   a) easy manipulation of the tensing devices
   b) tension guarantee
   c) use of low-cost dynamometric devices
2. Wire arrangements, making possible
   a) accurate covering, effective protection leading to eventual adherence
   b) friction reduction and friction control
   c) possibility to place special grids at the deviation points
3. Removal of the metallic ducts, making possible
   a) cost reduction
   b) no possible humidity in the injected ducts
   c) no slag possibly blocking wires when casting concrete
4. Anchorage
   a) metallic, making control quality easy
   b) standardized: single type for any case
   c) use of oval wires or with tolerance without great problems

d) possibility of partial prestressing
e) possibility of very thick cables
5. Beam with exterior cables
a) components prefabrication. Light components, prefabricated in plant
b) exterior cables, no inserting —cables are protected by any mortar layer or any other material (asbestos, etc…)
c) cables deviation by the means of hooks, thus cable profile adapted to any necessity, continuity or fitting being very easy[8]

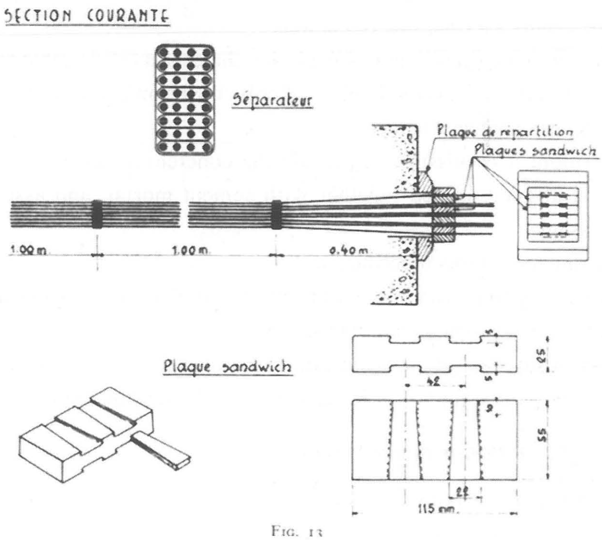


Figure 13
Different elements of the Magnel-Blaton system. (Magnel 1953f)

In the Freyssinet's procedure, the sandwich plates, figure 16, are substituted by a blocking system of a conical shape, figure 15, which applies tension to all the cable wires simultaneously, and thus requires very heavy strength from the jacks, figures 17 and 18.

In 1942, Magnel starts a campaign in favour of prestressed concrete that will later lead him as far as the United States and Canada where he gives conferences in front of assemblies of 700 people and he participates in TV shows.

Prestressing would always be his favourite research topic, both theoretical and experimental, together with laboratory tests. In 1953, two years before dying, he

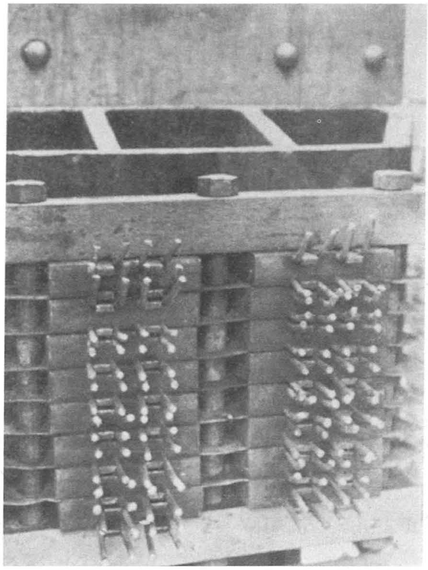
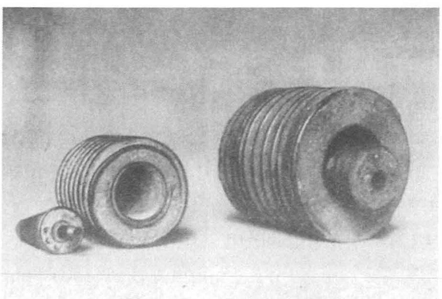
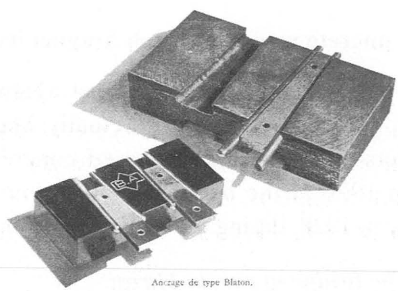


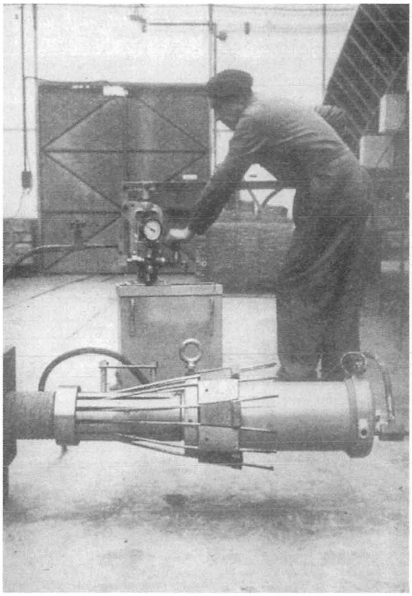
Figure 14
Several sandwich plates may be superimposed to form a more resistant cable. Here, 32 anchored arranged wires form a cable. (Magnel 1953f)

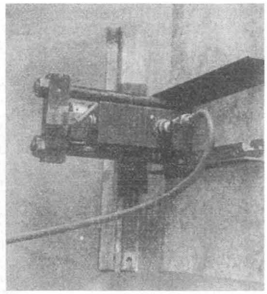
publishes an interesting paper on *Hyper staticity in prestressed concrete* (Magnel 1953b) in which he points out the necessity to provide engineers with a mathematical tool that will make possible the computation of such structures which can become very complex in case of prestressing.

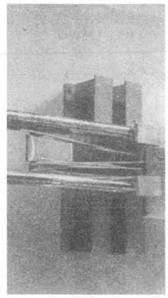




Figures 15 and 16
Freyssinet's anchorage (Magnel 1953f) and Magnel-Blaton's sandwich plate. (Robin 1951)







Figures 17 and 18
Freyssinet's jacks and those of the sandwich system. (Magnel 1953f)

The difficulties of calculation are the following: whereas in an isostatic system the application of a pre-stress to an element does not alter exterior reactions and thus does not create exterior bending moments, in an hyperstatic system the application of prestressing, by the linear and angular deformations that it generates, alters the exterior reactions and consequently creates exterior bending moments that we named "parasitic moments" (Magnel 1953b).

Concrete works in which Magnel had a contribution

The practice and expertise that Magnel tries to communicate to students surely comes from site work. Actually Magnel took part in the erection of several constructions in prestressed concrete, usually as a consultant, for example in 1926 at the beginning of the building of Koelkeberg basilica in Brussel, or in 1928, during the restoration of the Franciscan church in Saint Nicolas-Waas.

The bridge in rue du Mirroi

In spite of the war, Brussels carries on the construction of the North-South junction. It is a railway junction that connects the north and south stations. The bridge

in *rue du Miroir*, which is part of this junction, was built in 1943–1944. It is an oblique bridge that supports six railways. With the help of the National Funds for scientific research, the Belgian Railway Company and the National Office for the Junction, Magnel made it an experimental bridge, a part of which is the first prestressed concrete railway bridge in the world. He had the slab divided in six independent portions, one for each railway, and built them respectively one in plain concrete, two in concrete with reinforcements in Isteg steel, one in concrete with reinforcements in Toristeg steel, and two in prestressed concrete. The goal was to compare costs and quantities of material. Prestressed concrete, figure 19, was a little more expensive than the concrete with Isteg steel, but much thinner. The other two were more expensive and thicker.

Before building the bridge, a failure test was made with a beam having the same characteristic of a 65 cm segment of the bridge slab.

The success of this construction suggested the realization of the road bridges of Zammel (1944) and Eeklo (1945–1946), figure 20.

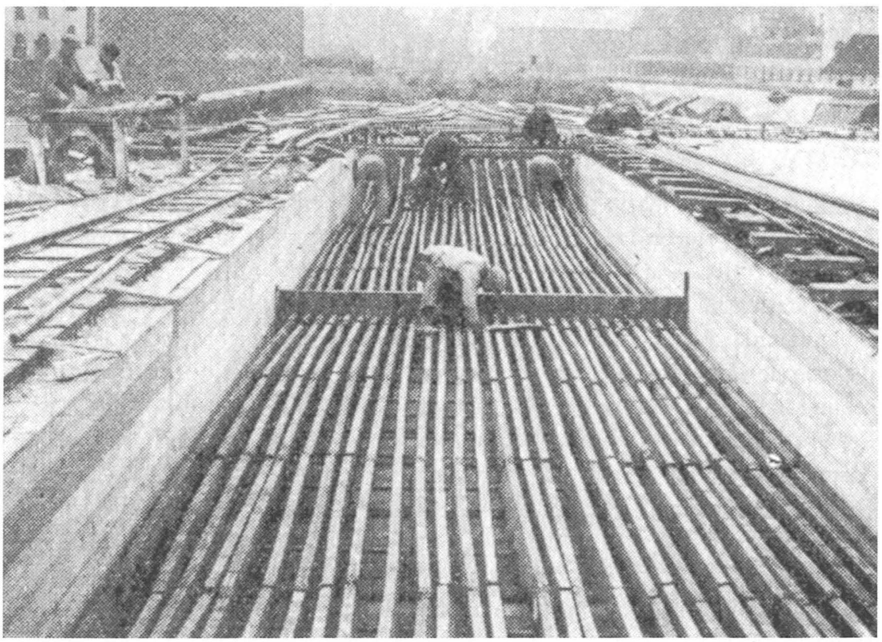


Figure 19
Prestressing cables before concreting the prestressed bridge of the *rue du Miroir*. (Magnel 1953f)

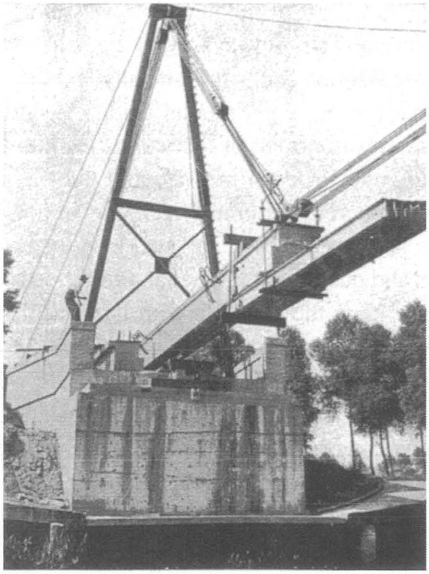


Figure 20
Positioning of the first beam of the Eeklo bridge. (Magnel 1949i)

Many other bridges will follow. Some with a personal contribution by Magnel, like the road bridge of Baudour, the Maghin bridge in Liège, the Malden bridge in Holland, and the Moerkerke bridge. Others will be built without his collaboration but using the Magnel-Blaton procedure, like the bridge in rue De Smet, and the W7 and W9 bridges near Ghent. Meanwhile, Magnel was particularly proud of his participation on the construction of two bridges that were built around the fifties. The first one is the Sclayn bridge, figure 23, over the Meuse, near Andenne, that was built by the firm Blaton-Aubert and is visible behind Magnel himself on figure 1. When it was first built, in 1949, this bridge was the first one in the world with continuous prestressed beams. It is 126 meters long, figures 21 and 22.

The second bridge is the Walnut Lane Bridge, figure 25, in Philadelphia, built in 1948–1949. It is the first prestressed bridge built in the United States. It is the result of Magnel's american campaign of 1946 in favour of prestressed concrete. In addition, during its construction, a failure test was made on a beam 47 meters long, figure 24. The test was executed in Magnel's laboratory in Ghent. This work surely is one of the feathers of Magnel's carrier.

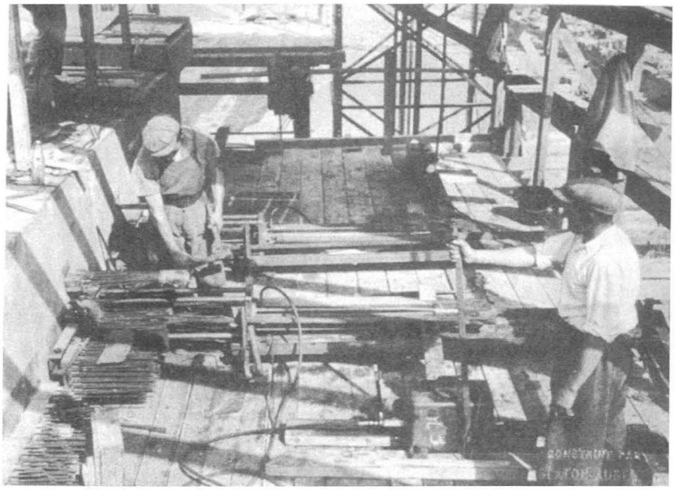


Figure 21
The application of prestressing to the Sclayn bridge. (Magnel 1953f)

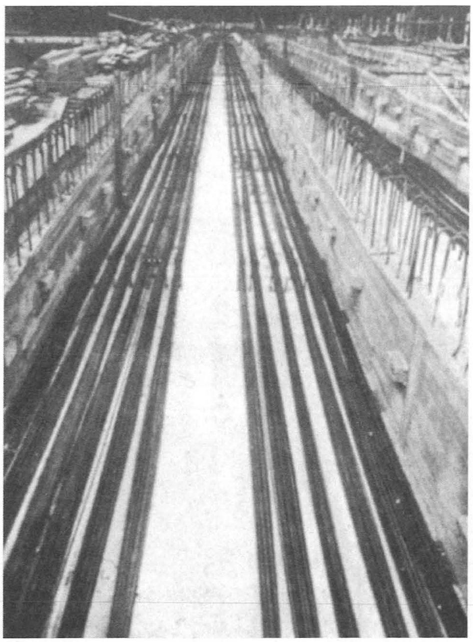


Figure 22
The prestressing cables of the Sclayn bridge. (Magnel 1953f)

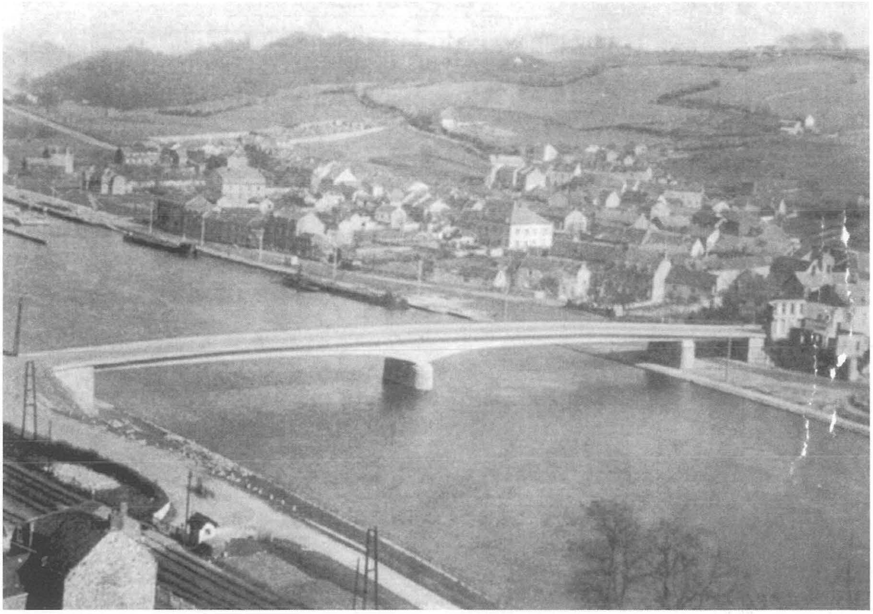


Figure 23
The Sclayn bridge. (Magnel 1953f)

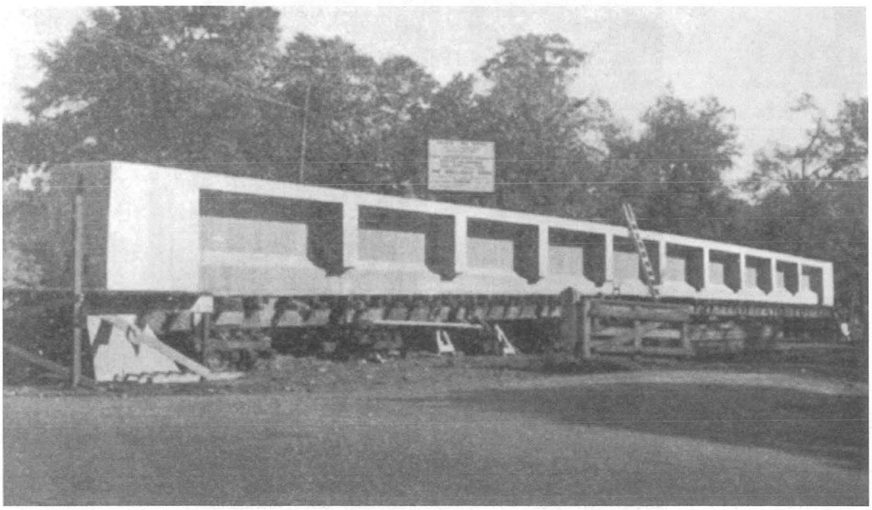


Figure 24
Test beam of the Walnut Lane Bridge. (Magnel 1953f)

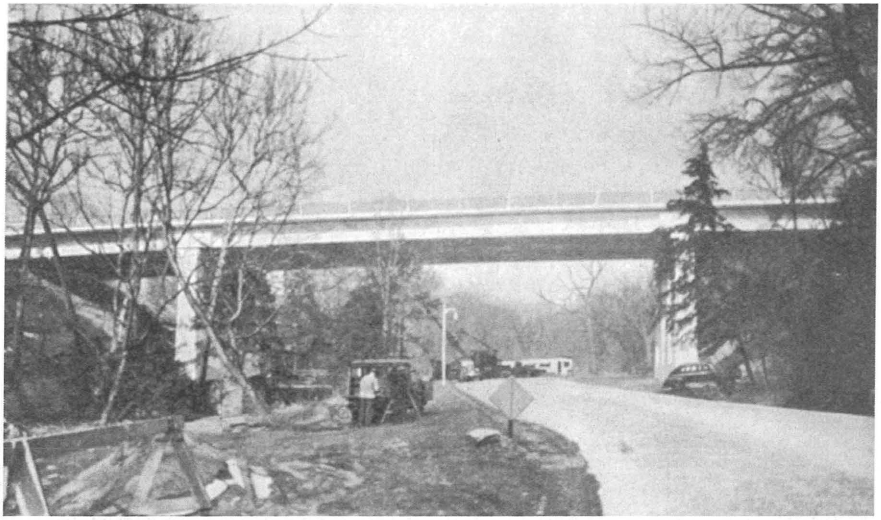


Figure 25
Walnut Lane Bridge. (Magnel 1953f)

Underpinning during restoration works

The tower of Saint Nicolas church, in Ghent, needed a whole and delicate restoration. In his preliminary studies, Magnel suggests to create two belt-beams in prestressed concrete, one of which to be put at the very bottom of the tower itself, i. e. at the top of the four pillars of the transept crossing. The construction works will be carried out after his death, and they will follow Magnel's suggestion, the necessity of which he had convincingly explained.

> We cannot temporarily put the tower on some supports, close to the actual supports that have proved to be so efficient for centuries.
>
> We must built first, towards the bottom of the tower, a solid support in reinforced concrete, in the shape of a rigid frame, hidden inside the thickness of the masonry. It is the construction of this frame —we might say "of this rectangular belt"— that requires the use of prestressed concrete, because its execution requires —just like a real underpinning— to demolish almost the full thickness of the peripheral walls of the tower, and this can be done only by emptying successive niches and by filling them with concrete as soon as they are dug. In this concrete we will place some thin tinny pipes that, all together, will form some hollow tubes running through the whole length of the four sides of the frame and make possible the insertion of the necessary cables for prestressing. (Magnel 1944c)

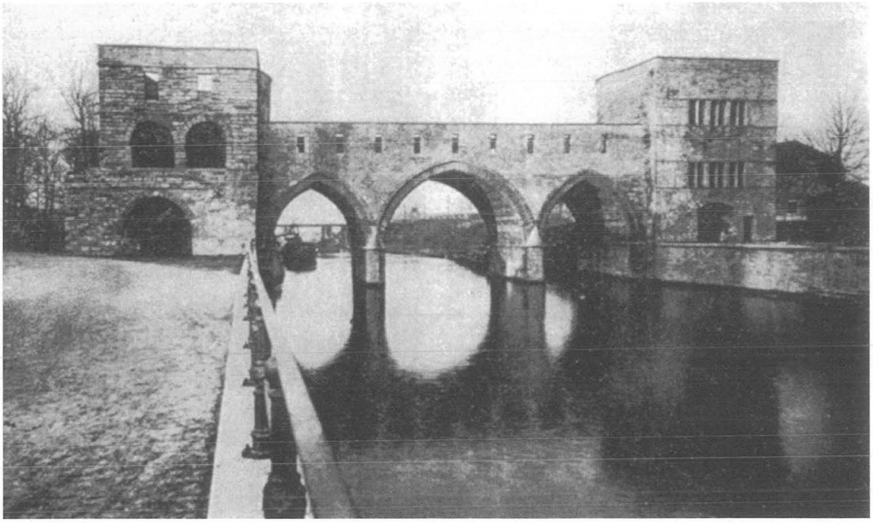


Figure 26
The *des trous* Bridge in Tournai. (Magnel 1953f)

A similar operation was done to raise the two towers of the *des Trous* Bridge in Tournai.

Other works

In 1947–48 Magnel collaborates in the construction of a new plant for the Cotton Union, figure 27, in Ghent. It is a building of 30,000 square meters on a single ground level. It is interesting to notice that the big size of this plant immediately suggested standardization and work division in modules.

> In order to save money, we decided to build the construction with as much standardization as possible
>
> . . .
>
> The building's design was based on a 3.60 m module, which suits best the dimensions of the machines that have to be installed (Magnel 1948f).

These concepts are found again in the planes hangars built in 1947 in Melsbroek, figure 28.

In 1951, when the International Prestressing Meeting is organized in Ghent to commemorate the 75th anniversary of the Association of Ghent Engineers, Magnel collaborates in the construction of a monumental staircase, figure 30, located

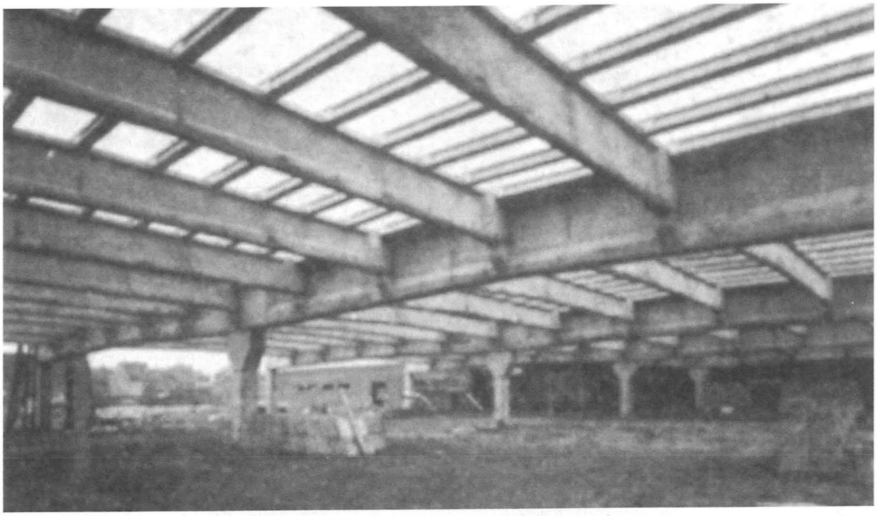


Figure 27
The prestressed beams of the Cotton Union plant's roof. (Magnel 1953f)

in the showroom of General Motors in Ghent. It is a spiral stair and it therefore undertakes torsion. Reactions to torsion being badly known, Magnel builds a similar full scale staircase, figure 29, in order to analyse its reactions when stressed and loaded.

He describes the construction of the staircase in a humorous paper that shows a Cadillac placed in the showroom and on the staircase itself.

> Cadillac: We have been looking at each other for two days now; why not make friends? You look just as aristocratic as I do. By the way, who is your father?
>
> Staircase: Can't you guess that? Don't you know we are brothers? Our father is the famous general —General Motors. As far as I know, we don't have the same mother.

Figure 28
Melsbroek planes hangars façade. (Magnel 1953f)

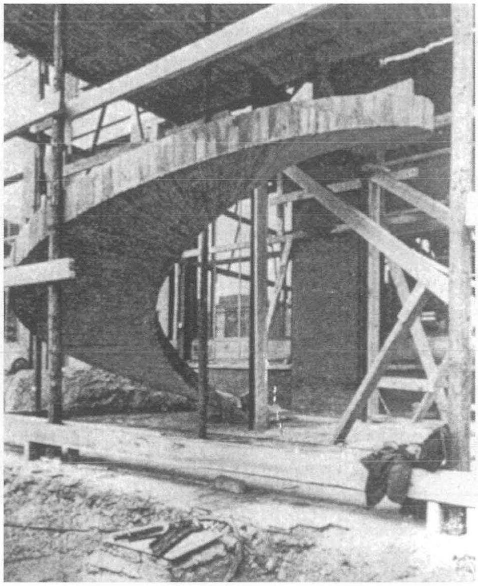


Figure 29
The testing staircase (Magnel 1953f)

As a matter of fact, from what I hear from conversations around me, I have several mothers, and my friends call them "the Architects".

Cadillac: I am thinking of the trouble the doctor must have had in your delivery, owing to your shape being unusual.

Staircase: It was indeed a very delicate and difficult operation by a doctor named Contractor. Since he was afraid of the intricate operation required in bringing me to daylight, he consulted a specialist called the Professor, who often comes here: he decided to make an experimental delivery in his laboratory in Ghent, of a staircase born of less aristocratic parents. The professor even tested the health of the child and gave useful advice to the doctor for my birth and the strengthening of my delicate body by an operation called pre-stressing. The professor said I could not live long without being prestressed. I must say to his credit that I feel much stronger now (Magnel 1951e).

The last project

Magnel had committed himself to a very important project for the Brussels Universal Exhibition in 1958: the telecommunication tower, figure 31, an awesome tower, 507 meters high, topped with an antenna of 200 meters.

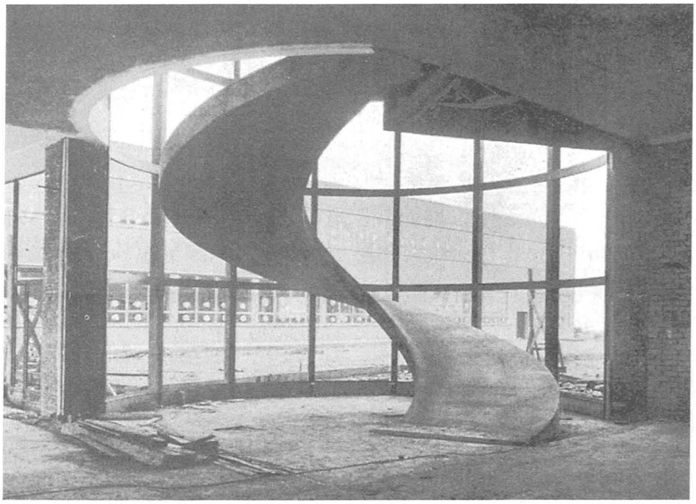


Figure 30
The concreted staircase of the General Motors showroom. (Robin 1951)

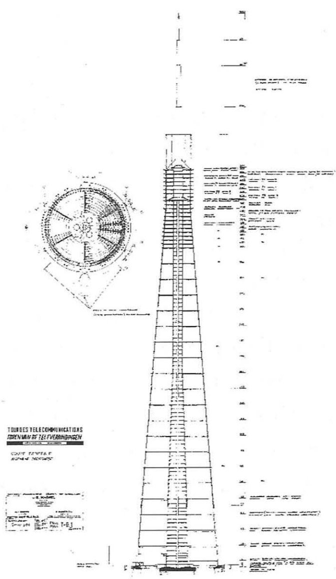


Figure 31
Telecommunication tower for the Universal Exhibition in 1958 (Magnel Laboratory Archives)

Unfortunately the tower was never built, and this was probably partly due to Gustave Magnel's death.

Conclusion

Even if he is not the inventor of prestressed concrete, as he himself used to admit, it seems fair to state that Magnel proved, throughout Belgium, the United States, Canada and other countries, how appealing this new building material is. He did so by giving lectures but also mostly by real practice. He collaborated in the realization of a great number of constructions in prestressed concrete. Furthermore he showed contractors the reliability of this material by running accurate tests in his laboratory in Ghent.

While studying Magnel's work we can't help to be struck by both the wide range of his interests and the coherence of his thoughts. His interests cover beams and bridges construction, metallic first, then Vierendeel, and after in prestressed concrete —since he immediately understood the advantages of this material— and also laboratory tests to ensure constructions' safeness and the institution, inside the SECO Bureau, of an insurance system for contractors who had to undergo ten-years guarantee. In addition to all this, he also was a professor and a researcher and he made innovations until the end of his life, for example by building this helical staircase. He was furthermore concerned by profit, and reduced the needed amount of construction material thanks to prestressing, and reduced the time required for computing by providing engineers with handy, quick and efficient calculus methods.

We thus can conclude that Magnel approached all the facets of the engineer's profession and did not neglect any of its aspects.

Notes

1. Special thanks to Mr. Y. Pianet, the actual director of the "*bureau SECO*", who allowed me into the SECO Archives, and also to Mr. A. Broucke and Prof. D. Vandepitte for the abundant information they gave me and for their careful reading of this text.
2. *Een kracht kan wel een vector zijn voor een wiskundige, maar voor de ingenieur is het iets dat duwt of trekt.*
3. *De professoren moeten ofwel hun cursussen uitgeven of het boek van een andere auteur aanbevelen. Het zou bijna de studenten moeten veboden worden nota's te nemen; als men een cursus moet opnemen kan men niet nadenken.*
4. This building is still extant.
5. These machines are in the entrance of the actual Magnel Laboratory in Zwijnaarde. I thank Prof. Luc Taerwe, the actual director of the Magnel laboratory, for his welcoming and for the huge documentation that he allowed me to consult.
6. These machines are in the entrance of the actual Magnel Laboratory in Zwijnaarde.

7. I wish to warmly thank Philipe Blaton for the efficacy with which he gave me important information about this patent.
8. This text comes from a document given to me by Philippe Blaton. I think it is very close to the patent text.

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Non-smooth formulation for the ancient and strongly non-linear problem of block dynamics

Anna Sinopoli
Alessio Ageno

The dynamics of a rigid block, free-standing with friction on a moving ground, has been the object of study in the international scientific community for a long time. The interest in this kind of problems lies in the possibility of understanding the seismic behavior of a variety of man-made structures which can be modeled as free-standing rigid blocks, such as monumental stone structures, concrete radiation shields, water towers, power transformers and any other free-standing equipment, including furniture such as art objects stored in museums.

The first investigations dating back to the end of the 19th century (Milne 1881; Milne 1885) were motivated by the opportunity and possibility of using information about stability with regard to the overturning of stone blocks, as this may be indicative of seismic intensity in places without seismographs. After more than a century, and a intense period of research in recent times, the question of stability with respect to the earthquake of these objects seems nonetheless to be a matter which bears further investigation.

The dynamics of such a system is in fact so complex that it can be taken as emblematic of themes emerging in modern non-linear dynamics: the unilateral character of the contact with the ground, the presence of Coulomb friction in the points maintaining contact, and the velocity discontinuities which occur each time the block impacts the ground have the effect that, during its dynamic evolution, the system is characterized by different kinds of discontinuity that can be manifested at the same instant, or in different instants. In particular, depending on the values of the geometric and mechanical parameters and the initial conditions, the system can evolve according to diverse mechanisms governed by different systems of differential equations.

The continuous dynamical phases are matched at given instants according to switching conditions. The impact instants are in particular those in which all kinds of discontinuities can occur at the same time; the velocities discontinuity due to the impact is governed by the law of inelastic impact.

Discontinuous systems of this kind are often called *switching* or *hybrid* systems (Leine 2000), since they reveal a mixed continuous and discrete nature.

The complexity of the dynamics of a free-standing block on a moving ground presents singular aspects which are part of different and specific research fields. There are formal aspects of theoretical completeness concerning the dynamic formulation: problems of this kind can be considered as a dynamical extension of linear complementarity problems and, in convex analysis, the corresponding dynamic equations can be formulated as differential inclusions (Moreau 1988; Glocker and Pfeiffer 1992; Sinopoli 1997).

On the other hand, there is another vast line of research in the scientific literature that, motivated by practical, engineering aims in connection with the need to evaluate the seismic safety of those systems, has assumed simplifications of the general model by means of an "a priori" reduction in the degrees of freedoms, i. e., by adopting the assumption of friction large enough to allow only rocking mode, and the use of restitution coefficients to take the variations in velocity due to impacts into account (Housner 1963; Spanos and Koh 1984; Hogan 1989; Yim and Lin 1991; Iyengar and Roy 1996; Virgin, Fielder and Plaut 1996; Plaut, Fielder and Virgin 1996). These works, starting with the pioneering paper by Housner (1963), have made it possible to obtain information about the features of the motion and its stability by means of analytical and numerical investigations. In particular, the stability analysis and a correct evaluation of stability indicators have been only recently performed by means of variational procedures adapted to these kinds of non-smooth systems (Sinopoli and Ageno 2001; Ageno and Sinopoli 2003; Ageno and Sinopoli 2005; Sinopoli and Ageno 2005).

Lastly, there is a third way of tackling the dynamic analysis of a free-standing block with friction on a moving foundation (Ishiyama 1982; Sinopoli 1987a,b; Sinopoli 1989c; Ageno and Sinopoli 1991; Sinopoli 1991; Shenton and Jones 1991a; Shenton and Jones 1991b; Augusti and Sinopoli 1992; Sinopoli and Sepe 1993; Sinopoli 1995b; Sinopoli 1997): this has been adopted by the authors who, without renouncing the complexity of the general formulation of the problem, are trying by means of analytical or semi-analytical methods to obtain information on the dynamic behavior of the system as a function of its mechanical and geometric parameters.

By assuming a previous critical review on the subject (Augusti and Sinopoli 1992) as its starting point, this paper shall analyze and discuss the complexity of the problem by emphasizing those fundamental aspects which are strictly related to recent results obtained.

First, the mechanical model is introduced: kinematics, unilaterality of contacts, consequent features of the equilibrium configuration and free dynamics are discussed. Forced rocking dynamics and multi-variety of responses is then introduced. Lastly, problems concerning both global and local stability analyses are outlined by tracing the lines of future research.

The problem

Let us introduce the problem and consider the motion of a rigid block (Fig. 1) simply supported on a moving ground. Assume a moving ground reference system (O,x,y), with which the unit vectors $(\boldsymbol{t}, \boldsymbol{n})$ are associated —$\boldsymbol{t}$ is parallel to the moving boundary Γ and $\boldsymbol{n}$ is oriented outwards from Γ—, and let (x_G, y_G, θ) be the Lagrangian coordinates of the plane motion, where G is the center of mass and θ the rotation angle; in addition, let M be the mass of the block, b and h the base and height length, respectively, and $a_H = \ddot{x} = \beta\,\alpha\,g\,\cos(\Omega\,\tau + \phi)$ a harmonic ground acceleration, with $tg\alpha = b/h$ and g gravity acceleration.

Since a unilateral constraint does not reduce the degrees of freedom, the differential equations governing the Lagrangian coordinates, during the phases of continuous motion, can be written as:

$$\begin{gathered} M\ddot{x}_G = \lambda_1 - M\ddot{x} \\ M\ddot{y}_G = \lambda_2 - M g \\ I_G\,\ddot{\theta} = \frac{R}{2}\,[\lambda_1 cos(\alpha - |\theta|) - sgn\theta\,\lambda_2\,sin(\alpha - |\theta|)] \end{gathered} \tag{1}$$

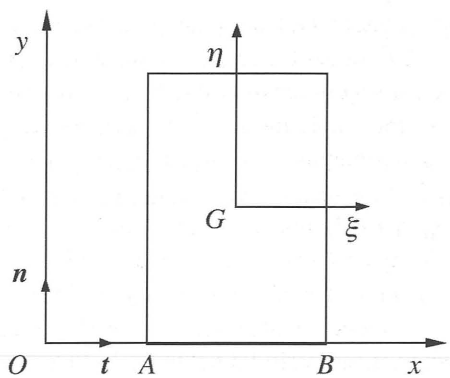

Figure 1
Rigid block on a horizontally moving rigid ground

where R is the half-diagonal of the block, I_G the inertia moment, and λ_1 and λ_2 the horizontal and vertical reactions due to the contact with the ground; they are governed by Coulomb's law and the contact law of impenetrability, respectively. Even taking into account the case in which the block is initially at rest, the activation, either at the beginning or during the dynamic evolution of one, two or three of the system's equations (1), according to a given mechanism, depends on the geometric size of the block (namely, b and h), on the value of the friction coefficient μ, on the instantaneous value of the forcing acceleration, and mainly on the effects of the impacts which occur every time some point of the block comes suddenly into contact with the ground. Thus, phases of resting, rocking or sliding-rocking around the base corners, sliding or up-lift are possible, and the structure of the differential equations changes since specific analytical expressions of the ground reactions λ_1 and λ_2 correspond to each activated mechanism.

General dynamics formulation

Consider first the case of absence of contact (uplift). Thus, the constraints are ineffective; both λ_1 and λ_2 are equal to zero and the dynamics is free. However, this occurrence is unlikely in the presence of a horizontal ground motion and inelastic impacts so that it will be no more discussed here.

Unilateral and friction contact laws

In the case of contact, the ground reactions λ_1 and λ_2 are among the unknown quantities of the problem; their determination follows from the solution until convergence of two sub-problems, the normal and tangent sub-problem, respectively, which are each other connected.

The normal problem, governed by the impenetrability law, allows only bounded variations for the position of any contact point P of the base AB of the block (Fig. 1); thus, only normal relative velocities $\dot{y}_P = \dot{\boldsymbol{r}}_P \cdot \boldsymbol{n} = \dot{r}_{P,n} \geq 0$ are admissible. During the phases of either smooth or non-smooth dynamics a finite or impulsive, respectively, reaction $\lambda_2 = R_{P,n} \geq 0$ is presumed to act to prevent penetration and the normal contact law assumes the Signorini well-known expression of a linear complementarity problem:

$$R_{P,n} \cdot \dot{r}_{P,n} = 0 \qquad \text{with:} \quad R_{P,n} \geq 0; \quad \dot{r}_{P,n} \geq 0 \tag{2}$$

The tangent problem is governed by Coulomb's friction law. Thus, during the phases of either smooth or non-smooth dynamics, tangent and normal reactions —either finite or impulsive— must satisfy the following inequalities, in dependence on the value of the relative tangent velocity $\dot{r}_{P,t}$:

$$|R_{P,t}| \leq \mu \, R_{P,n} \qquad \dot{r}_{P,t} = 0$$
$$R_{P,t} = -\mu \, sgn(\dot{r}_{P,t}) R_{P,n} \qquad \dot{r}_{P,t} \neq 0 \tag{3}$$

where $\lambda_1 = R_{P,t}$ and μ is the friction coefficient. The tangent contact law therefore becomes:

$$R_{P,t} \cdot \dot{r}_{P,t} \leq 0 \tag{4}$$

with $R_{P,t}$ governed by (3) and by the law of maximum dissipation of power.

In the case of non-smooth dynamics, relationships (2)–(4) hold for right-sided velocity values since the velocity exhibits a discontinuity.

The general dynamic formulation is now possible by taking the contact laws into account. Numerical methods are generally adopted when dealing with the dynamics of rigid bodies subject to Coulomb friction and unilateral constraints. The dynamics is then formulated as a linear complementarity problem in mathematical programming (e. g.: Lötstedt 1982; Jean and Moreau 1992; Glocker and Pfeiffer 1992). In this context, in the presence of friction, the sufficient conditions for existence and uniqueness of the solution were given by Lötstedt (1981), who demonstrated that the conditions corresponding to the fulfilment of Coulomb law are equivalent to the Kuhn-Tucker conditions satisfied by the optimal solution of a quadratic programming problem. Thus, under appropriate assumptions concerning the position of the system, the problem corresponds to the minimization of a convex function over a convex set implying the existence of a unique solution.

However, due to the presence of Coulomb friction, the identification of the starting mechanism can sometimes seem unclear, as in the case of forced sliding-rocking pointed out by Shenton and Jones (1991a). This is yet another question to add to those where, due to friction, it seems that either no solution exists or, if it does, it is not necessarily unique. Since the beginning of this century many authors have discussed this fact; e.g. Painlevé 1895; Painlevé 1905a; Painlevé 1905b; Béghin 1923–24; Béghin 1924–25; Delassus 1920; Pérès 1953.

This is one among the reasons that have led to the introduction of a dynamic formulation which makes it possible to examine phases of smooth or non-smooth dynamics and determine both the activated mechanism and instantaneous acceleration (Sinopoli, 1997). The basic concepts of this formulation will be synthetically presented in the following, together with the first result obtained.

Dynamic formulation and geometric representation in the configuration space

By taking the contact laws into account, the dynamic formulation proposed by Sinopoli (1997) has been performed by introducing a parametrization: the dynamic evolution of the system has been thus studied as the trajectory of its representative point in the configuration space, where the metrics of the kinetic energy is adopted to guarantee the Euclidean structure of the space.

This formulation allows a geometric representation in the configuration space, which permits us to visualize the instantaneous dynamic situation, particularly in those cases in which it seems unclear because of friction, and therefore improve our understanding of the problem. The geometric representation is more clear in the plane of the normal contact; that is, in the plane associated to the angular velocity $\dot{u}_\vartheta$ of the block and the normal velocity $\dot{u}_y$ of the center of mass, which are governed by the normal contact law.

In figure 2, which corresponds to the block at rest, two convex cones bounded by corresponding couples of unit vectors are shown. In accordance to the normal contact law, (t^n_A, t^n_B) and (n^n_A, n^n_B) are the sets of admissible normal velocities and reactions, respectively, for contact at any point of the base of the block with boundary positions at either point A or B.

Figure 3 refers, during the dynamic evolution of the block, to the case of rocking with contact at either corner edge of the base; there, the generalized Coulomb

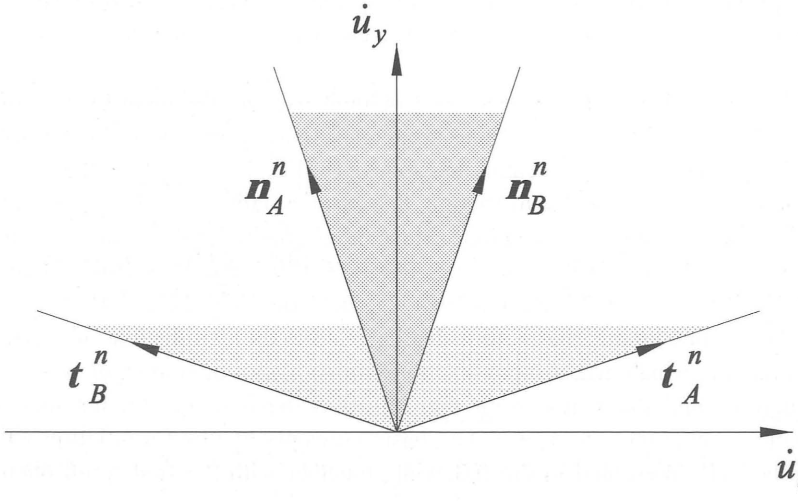

Figure 2
Convex domains of generalized admissible normal velocities and reactions

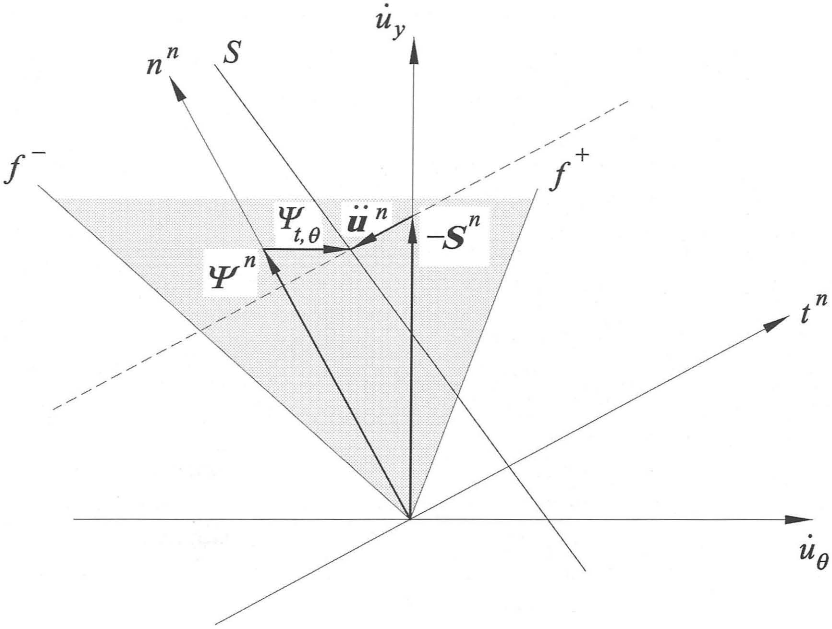

Figure 3
Coulomb cone and dynamic balance for contact at either corner edge

domain, bounded by the two limit half-lines (f^+, f^-), is shown. As it is possible to recognize, the dynamic situation corresponds to a balance equation (closure of corresponding vectors polygon) between generalized acceleration $\ddot{\boldsymbol{u}}^n$ and forces; $\boldsymbol{S}^n$ is the generalized active force, whereas $\boldsymbol{\Psi}^n$ and $\boldsymbol{\Psi}_{t,\theta}$ are the generalized normal and tangential reactions, respectively. Contact at the given point can be maintained with motion characterized by rocking without sliding since an acceleration $\ddot{\boldsymbol{u}}^n$ exists parallel to the boundary $\boldsymbol{t}^n$ of the contact point and crossing at point P —inside the Coulomb cone— the sticking line of the contact point (Sinopoli 1997).

Starting mechanisms for the block dynamics

Let us illustrate now a first analytical result obtained by the formulation above. Consider the case when the block is initially at rest; assume that $K_s = \beta\ \alpha\ cos\phi$ is the initial horizontal acceleration of the ground in g units and analyze the starting mechanism, as a function of K_s, b/h and the friction coefficient μ.

Initially, there is contact along the whole base AB of the body; thus, the unilateral constraint bounds the velocity of all the points in contact, and in the plane of generalized velocities $(\dot{u}_\vartheta, \dot{u}_y)$ the tangent domain is the closed and convex set defined by: $\dot{r}_{P,n} \geq 0$, $\forall P \in AB$. The boundary of this domain is defined by the two half-lines associated with the unit vectors $\boldsymbol{t}^n_A$ and $\boldsymbol{t}^n_B$; they correspond to the directions defined by $\dot{r}_{A,n} = \boldsymbol{n}^n_A \cdot \dot{\boldsymbol{u}} = 0$ and $\dot{r}_{B,n} = \boldsymbol{n}^n_B \cdot \dot{\boldsymbol{u}} = 0$, respectively (Fig. 2). In the plane $(\dot{u}_\vartheta, \dot{u}_y)$, the active force, only due to self-weight, is $\boldsymbol{S}^n \equiv (0, -\sqrt{M}\, g)$. The normal generalized reactive force $\boldsymbol{\Psi}^n$, which arises from the unilateral constraint and can be defined only as resultant, acts at any point of edge AB. $\boldsymbol{\Psi}^n$ belongs to the normal domain defined as the closed and convex set, normal to the tangent domain and bounded by the half-lines associated with the unit vectors $\boldsymbol{n}^n_A$ and $\boldsymbol{n}^n_B$; they correspond to contact at either point A or B, respectively (Fig. 2).

A Coulomb's domain, depending on the value of the friction coefficient μ and a sticking line s are associated with each of vectors $\boldsymbol{n}^n_A$ and $\boldsymbol{n}^n_B$; in figure 4 they are shown for small values of the friction coefficient μ.

Contact will be maintained at either point A or B if rocking or sliding-rocking is activated, whereas contact will be along the whole edge AB in the case of resting or sliding.

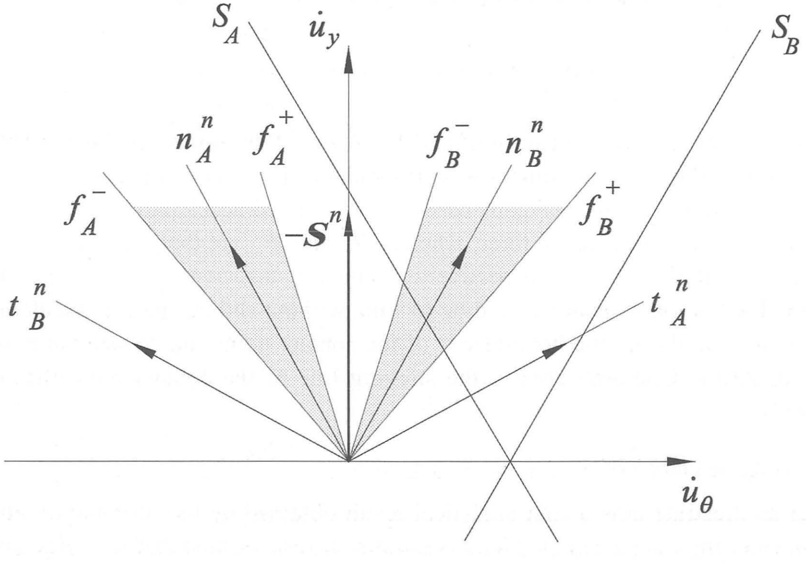

Figure 4
Coulomb cones and sticking lines for small values of friction

To identify the starting mechanism assume $K_s > 0$; thus, contact can be maintained at either point A or along edge AB. Moreover, since the initial velocity $\dot{\boldsymbol{u}}^n$ is zero, rocking or sliding-rocking with contact at point A can start only if an acceleration $\ddot{\boldsymbol{u}}^n$ exists, parallel to $\boldsymbol{t}_A^n$ and accordingly oriented —that is, if $K_s > b/h$—, and the friction coefficient μ is large enough so that f_A^+ is beyond the $\dot{u}_y$ axis — that is, if $\mu > b/h$.

In this case, rocking will be activated, if the intersection point P (Fig. 3) between acceleration $\ddot{\boldsymbol{u}}^n$ and sticking line s_A^n belongs to the Coulomb's domain, that is, if:

$$\mu \geq \frac{3(b/h) + K_s[1 + 4(b/h)^2]}{3K_s(b/h) + 4 + (b/h)^2} \tag{5}$$

otherwise, sliding-rocking is activated, with $\ddot{\boldsymbol{u}}^n$ certainly belonging to the tangent domain because $\mu > b/h$ (Sinopoli 1997).

Note that $K_s = b/h$ represents the minimum value of the excitation required to start rotation around the base corner edge; this value is referred here as *seismic threshold* and his mechanical meaning will be more clearly discussed in the following.

For values of μ lower than (which is the case of figure 4) or equal to b/h, the only possible value for the generalized acceleration is $\ddot{\boldsymbol{u}}^n = \boldsymbol{0}$; thus, contact is maintained along all the base and sliding or resting depends on whether $\mu < K_s$ or not. The regions corresponding to the starting of a particular mechanism, as a function of K_s, μ and b/h , are shown in figure 5. Each mechanism is labeled by means of the initial letter of the corresponding motion; then, R means rocking, S sliding, *Re* block at rest and *S-R* sliding-rocking.

Inelastic impact of the block

The formulation above can be extended also to phases of non-smooth dynamics, especially in the case when, coming from rocking mode, the blocks enters suddenly into contact with the ground; thus, the velocities distribution violates the unilateral contact law and an impact occurs. This situation like that of the previous section corresponds to an extended contact along the side AB of the block.

Assume inelastic impact; this means that the impact ends as soon as the velocity distribution of contact points becomes admissible so that either an unknown corner edge or whole the base will remain in contact with the ground (no bouncing). As a consequence the normal generalized velocity after the impact $\dot{\boldsymbol{u}}^{n^+}$ belongs to the boundary of the tangent domain (t_A^n, t_B^n).

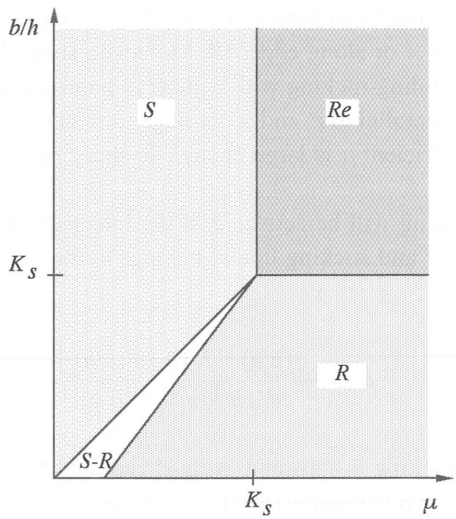

Figure 5
Regions of starting mechanisms in the plane of parameters

Impulsive reactions are responsible of the velocity discontinuity required to reach the boundary of admissible generalized velocities (Sinopoli 1997; Sinopoli 1998).

Let us illustrate here the case of frictionless impact as a function of the dimensional ratio *b/h*. According to the normal contact law (2), a linear-complementarity law relates normal velocity after the impact $\dot{\boldsymbol{u}}^{n^+}$ and velocity variation $\Delta\dot{\boldsymbol{u}}^n$:

$$\Delta\dot{\boldsymbol{u}}^n \cdot \dot{\boldsymbol{u}}^{n^+} = 0 \tag{6}$$

with $\dot{\boldsymbol{u}}^{n^+}$ belonging to the boundary of the domain (t^n_A, t^n_B) and $\Delta\dot{\boldsymbol{u}}^n$ belonging to domain (n^n_A, n^n_B).

Thus, only two situations are possible:

a) $\dot{\boldsymbol{u}}^{n^+}$ is different from zero vector; it belongs to either boundary of the tangent domain and belongs to the associated boundary of the normal domain (Fig. 6a);

b) $\dot{\boldsymbol{u}}^{n^+}$ is zero and $\Delta\dot{\boldsymbol{u}}^n$ is internal to the normal domain (Fig. 6b).

The occurrence of either situation (a) or (b) depends on the velocity before the impact and the geometric features of the system. In fact, assuming rocking or sliding-rocking before the impact, the case (a) describes the behavior of a slender

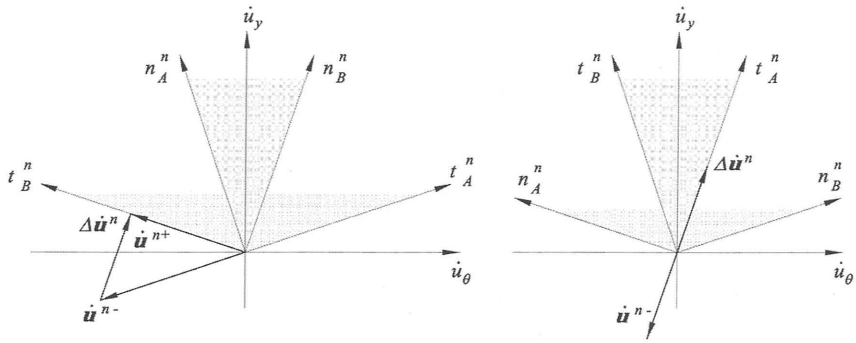

Figure 6
Inelastic frictionless impact: a) slender block; b) stocky block

block, which, coming from rotation around a corner edge, will rotate around the other corner edge after the impact; the situation (b), on the contrary, corresponds to the case of a stocky block, which can either slide or remain at rest after the impact.

Eq. (6) coincides with the formulation given by Moreau (1983) for standard inelastic impacts; but, it coincides also with the results obtained by Almansi (1916), who investigated the frictionless impact as a dynamics problem with unilateral constraints.

For impacts in the presence of Coulomb friction, an evolutive method can be adopted to avoid incorrect energetic balance; it is a generalization of that suggested first by Routh (Pérès 1953; Levi-Civita and Amaldi 1974). Again, two Coulomb cones and two sticking lines can be defined, corresponding to contact at either point *A* or *B*. Different solutions are then obtained, depending on initial conditions and both geometrical and mechanical features of the block.

In figure 7, the results of a parametric investigation (Sinopoli 1998) performed to identify the mechanisms activated in the post-impact motion are shown, as a function of the ratio *b/h* and the friction coefficient μ. Only the case of pre-impact rocking has been considered, while the notation is that of figure 5.

For values of μ appropriately large and *b/h* lower than $\sqrt{2}$, a wide region of rocking can be identified. This region and its boundary defines the validity of the impact model proposed by Housner (1963) for inverted pendulum-like structures. The separation value of $b/h = \sqrt{2}$ coincides with the result of a preliminary paper (Sinopoli, 1987), where the dry friction was roughly simulated.

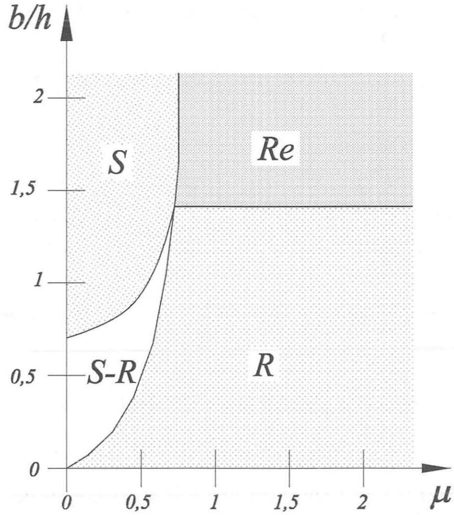

Figure 7
Mechanisms activated after the impact of a rocking block

Rocking mode dynamics

As discussed in previous sections, the dynamics of a block simply supported on a moving ground with friction can be characterized by the activation of different mechanisms during the dynamic evolution, so that the dynamic analysis of such a system can be very complicated. Nevertheless, from a practical applications point of view, the collapse of this kind of systems is strictly related to overturning so that the rotation angle can be considered as the critical dynamical parameter. In order to investigate its role, let us ignore the influence of dry friction in relation to sliding of the contact point; assume that the friction is large enough in order to allow only rocking mode and consider both free and forced dynamics.

Unforced dynamics: Mechanical features and potential energy

The free dynamics of a block during rocking modes is equivalent to that of two inverted *pendulum* simply supported on oblique planes (Figs. 8a, b). Since the vertical position of an inverted pendulum is non-stable, starting from the resting position both anti-clockwise and clockwise rotations of the block admit two attractors: actually, they are the resting and overturning attractor, respectively, each of which corresponds to the vertical and stable configuration of a direct

pendulum. In figure 9a and 9b, respectively, the features of the potential energy of the system are shown.

Consider first figure 9a, for anti-clockwise rotations: a) rest position is a boundary-imposed stable configuration whose attractor is a not-allowed stable direct pendulum configuration; b) overturning position is a boundary-imposed physically unstable configuration whose attractor is a not-allowed stable direct pendulum configuration; c) the maximum of the potential energy is a non-stable configuration whose attractors are both the resting and overturning positions. A similar situation can be observed in figure 9b, for clockwise rotations. Thus, each rocking mechanism of the system has two direct pendulum-like attractors; the

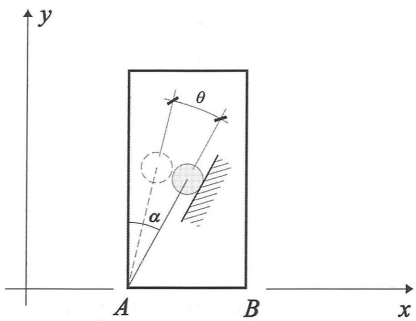

Figure 8a
Oblique pendulum for positive angles

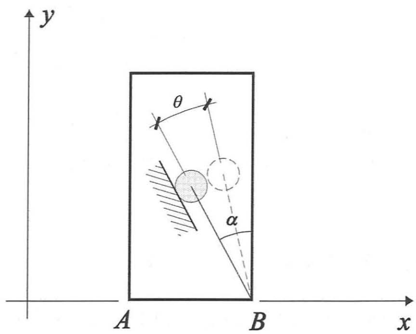

Figure 8b
Oblique pendulum for negative angles

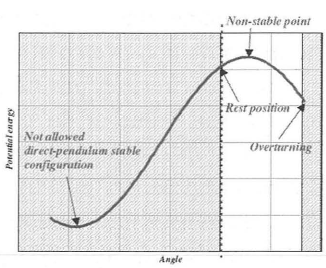

Figure 9a
Potential energy for positive angles

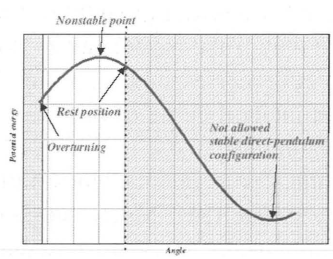

Figure 9b
Potential energy for negative angles

presence of boundaries, however, makes stable the resting attractor and unstable the overturning one.

The integral curves of figures 10 show the paths of the free motion, as a function of initial conditions, and the potential wells attracting the system either from resting or from overturning. Obviously, the stability of the free motion (no-overturning) depends on the energetic balance at the initial instant; if the motion starts along any path below the separatrix curve, it will reach asymptotically in the future the resting position by dissipating energy at each impact. On the contrary, if initial conditions determine a path along the separatrix curve the system can either come back to resting or reach overturning.

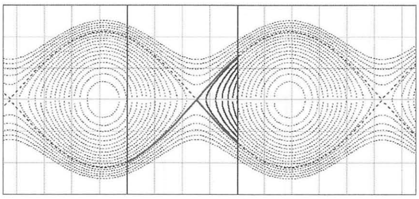

Figure 10a
Integral curves for positive angles

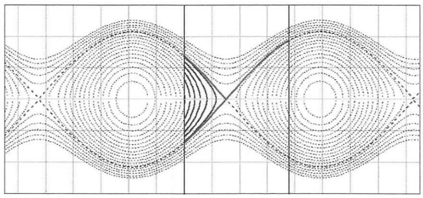

Figure 10b
Integral curves for negative angles

Observe also that stable oscillations of the system around the resting position are, in fact, oscillations of large amplitude around the stable equilibrium configuration; thus, free single oscillation dynamics is non-linear and the period depends on the oscillation amplitude (Fig. 11) as first observed by Housner in 1963.

Forced dynamics: Threshold and features of the responses

Consider now forced motion and harmonic ground excitations. The presence of potential wells and ground boundaries implies that rocking motion can be activated only if the instantaneous value of the external excitation overcomes a certain value corresponding to jumping of the system from resting position to a given integral curve of the phase portrait (Fig. 10). This value of the external excitation ($K_s > b/h$ in g units) is named here *seismic threshold* with in mind earthquakes as possible causes of motion and pioneering papers by Milne and Perry (Milne 1881; Milne 1885; Perry 1881) who tried to have information about seismic peaks in areas without instruments by observing still resting or overturned blocks.

However, due to the mechanical features of the system, it is quite impossible to obtain deterministic information both about seismic peaks from overturning of

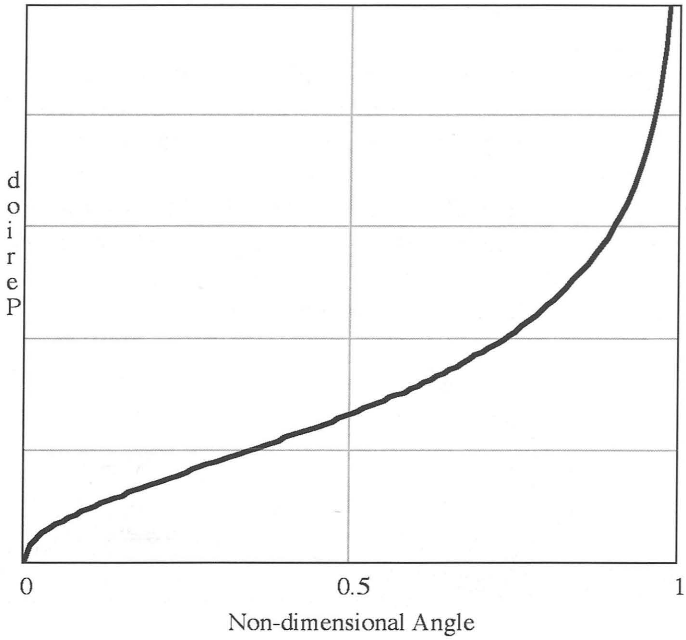

Figure 11
Relationship period-amplitude for free oscillations

blocks of given sizes and about overturning of given blocks from given features of earthquakes.

The problem is that the system even in the case of bounded motions is characterized by a multi-variety of responses due to the existence of potential wells.

In figure 12, regions of existence of different sub-harmonic responses can be observed, in the excitation parameters plane. β and ω are the non-dimensional excitation amplitude and frequency, respectively, and symbols like as *(1, n)* mean sub-harmonic responses with *1* impact for hemi-period of the response and *n* periods of the excitation in one response period. As shown, multiple responses —even if corresponding to different initial conditions— can coexist for equal values of the excitation parameters.

As enforcing confirmation of the results shown in figure 12, observe the curves in the phase portrait of figure 13. These dynamical responses have been obtained for same values of the excitation parameters, even if with different initial conditions. The intersection points of different curves, characterized by different periods and impacts per period (different colors), demonstrate the possibility to obtain different dynamical responses if the values of intersection points are

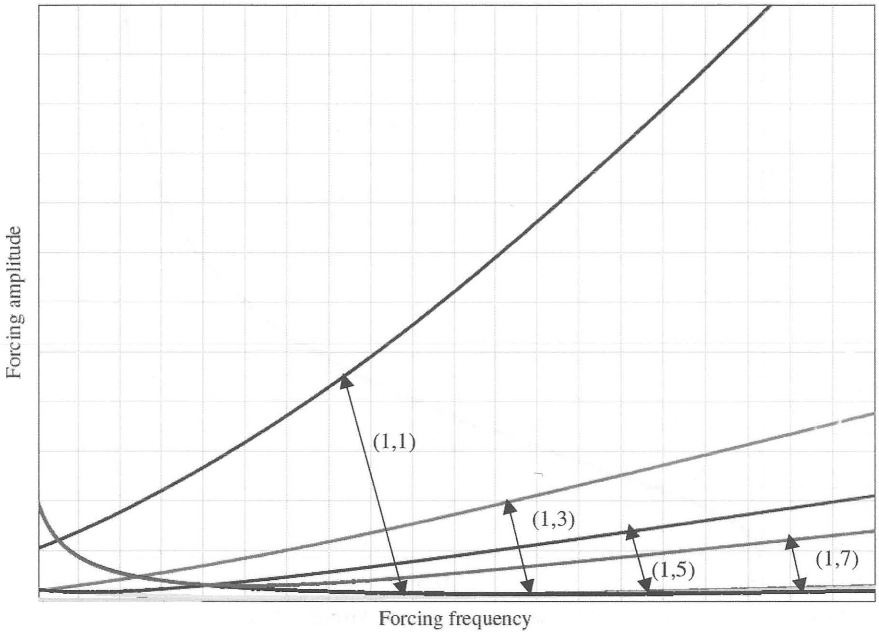

Figure 12
Regions of existence of (1,n) sub-harmonic responses in the excitation parameters plane for *b/h* = 1/4

assumed as initial conditions. Thus, the motion of the system, even if bounded, exhibits unpredictability features.

The multi-variety of behavior is still present if the system moves close to the non-stable equilibrium configuration (saddle point); there, the occurrence of overturning is governed by the attraction of two fixed points (potential wells) and, due to ground boundaries, the corresponding stable resting and unstable overturning configurations. Therefore, unpredictable too overturning behavior will be expected, as demonstrated first by Plaut, Fielder and Virgin (1996), who identified a fractal structure of the overturning boundary in the parameters plane.

Stability analysis

The stability analysis of the motion concerns the possibility to identify global (with respect to overturning) and local (with respect to given periodic response) stable motions. Generally, the global stability analysis gives only necessary

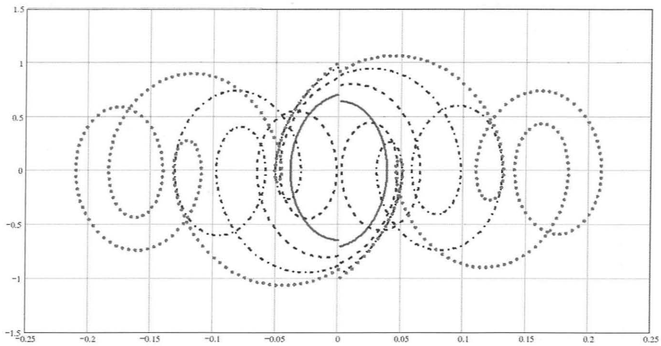

Figure 13
Phase portrait of different sub-harmonic responses with several intersection points

conditions for stable behaviors and, consequently, the corresponding curves lie far enough from actual significant values of the excitation parameters in the corresponding plane. Local stability analysis with related perturbative methods requires adaptations to this kind of non-smooth systems of standard procedures generally adopted in the case of smooth systems. The regions of stable sub-harmonic responses shown in figure 12 are just the results obtained by the adaptation of standard procedures of perturbative analysis to the case of forced dynamics of the block (Muller 1995; Ageno and Sinopoli 2005; Sinopoli and Ageno 2005).

By means of these methods quasi-periodic and period-doubling cascade behaviors have been identified; see as an example figure 14.

Research on the possibility of identifying the separation between deterministic and unpredictable behavior of the system, in the case of harmonically forced rocking mode, is still in progress. A first suggestion seems to be the identification of conditions to originate a-symmetric responses; the curious result, as shown in figure 15, is the occurrence of symmetry features in the activation of symmetry-breaking behavior.

Conclusions

The main critical points arising when the problem of modeling the dynamics of a rigid block simply supported on a moving rigid ground have been presented. Although the matter has been the subject of much research since the last century, this is a problem which still needs investigating.

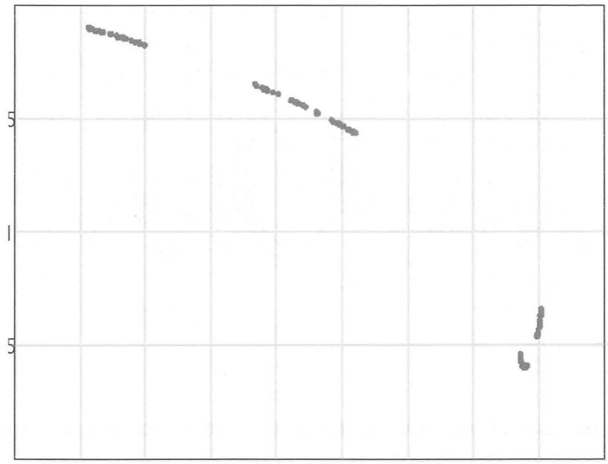

Figure 14
Chaotic attractor for $b/h = 1/4$, $\beta = 1.5338$ (forcing amplitude) and $\Omega = 1.5$ (forcing frequency)

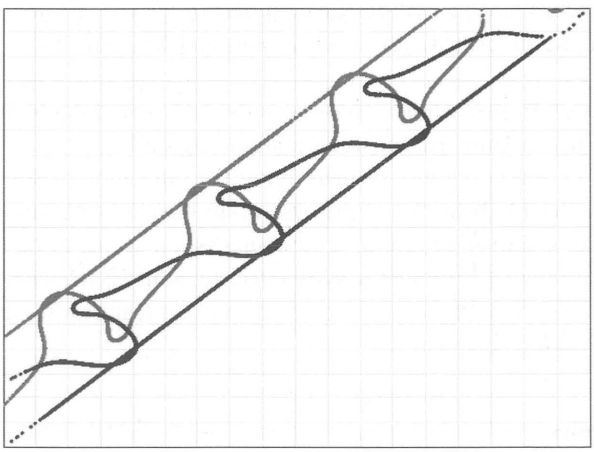

Figure 15
Intersections of the two curves mark the symmetric and asymmetric solutions in the plane of initial and first impact instants

The dynamics of such a system is in fact so complex that it can be taken as emblematic of themes emerging in modern non-linear dynamics: unilateral character of the contact, presence of Coulomb friction, velocity discontinuities during impacts and transition from one mechanism to another are all subjects that require non-smooth formulation for the dynamics.

In addition, the unilaterality of contacts implies features of the equilibrium configuration such that a multi-variety of responses characterizes the dynamic behavior of the system also in the case when a single mechanism is activated. The consequent unpredictability suggested the trace of future research concerning the identification of the on-set of a-symmetric responses as first breaking of deterministic behavior.

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